Qualitative Investigations and Approximate Methods for Impulsive Equations

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Nova Science Publishers, Inc.
New York

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Web Site: http://www.novapublishers.com

Library of Congress Cataloging-in-Publication Data
Hristova, Snezhana G.
Qualitative investigations and approximate methods for impulsive equations / Snezhana G. Hristova.
p. cm.
Includes bibliographical references and index.
ISBN978-1-61470-722-6 (eBook)
1. Impulsive differential equations. I. Title.
QA377.H75 2009
515'.353–dc22
2009004503

Published by Nova Science Publishers, Inc. ✜ New York

Contents

Preface

Although impulsive systems were defined in the 60's by Millman and Mishkis ([97], [98]), the theory of impulsive differential equations started its rapid development in the 80's and continues to develop today (monographs [16], [18], [89] and the cited therein bibliography).

The development of the theory of impulsive differential equations gives us an opportunity for some real world processes and phenomena to be modeled more accurately. Impulsive equations are used for modeling in many different areas of science and technology (see, for example, [44], [115]).

This book gives the idea of the wide specter of the theory of impulsive equations to the theoretical and applied researchers in mathematics, and provides them a tool and motivation for deeper investigations and applications of these equations to real world problems.

In the following book a wide class of impulsive equations are investigated such as:

– impulsive ordinary differential equations (scalar and n-dimensional case; with fixed and with variable moments of impulses);

– impulsive differential-difference equations (scalar and n-dimensional case; with fixed and with variable moments of impulses);

– impulsive functional-differential equations (n-dimensional case with fixed moments of impulses);

– impulsive hybrid differential equations (scalar and n-dimensional case; with fixed and with variable moments of impulses);

– impulsive differential equations with "supremum" (n-dimensional case with variable moments of impulses).

For the above mentioned impulsive equations some qualitative properties of the solutions are studied and approximate methods are applied. Various types of problems, such as initial value problems, periodic boundary value problems and linear boundary value problems for impulsive equations are considered.

In chapter 1 a systematic development of the theory of impulsive integral inequalities is presented. Various types of impulsive integral inequalities of Gronwall-Bellman and Bihari types are solved and some of their applications are given.

In chapter 2 the boundedness and the periodicity of the solutions of impulsive equations are studied. The investigations are made with the help of modifications of Lyapunov's method and Razumikhin method. The classical continuous Lyapunov's functions are commonly used for qualitative investigation of various types of differential equations without impulses ([37], [56], [84], [85], [86], [87], [96], [122], [123]). Since the solutions of impulsive equations are piecewise continuous functions, it is necessary to use an appropriately defined piecewise continuous functions that are similar to the classical Lyapunov's functions. It is noted that many authors apply piecewise continuous Lyapunov's functions to study the stability of the solutions of impulsive equations ([3], [57], [100], [113], [114], monographs [18], [89], and the cited therein bibliography). In this book piecewise continuous Lyapunov's functions are applied to study boundedness and periodicity of the solutions of various types of impulsive equations.

In chapter 3 and chapter 4 two approximate methods for solving impulsive equations are presented.

In chapter 3 monotone-iterative techniques are applied to different types of impulsive equations. The main characteristic of the considered monotone-iterative techniques is the combination of the method of lower and upper solutions and an appropriate monotone method. These techniques are applied successfully to different types of differential equations without impulses ([36], [92], [94], [102], [116], monograph [88], and the cited therein references). In this book algorithms for approximate finding of successive approximations to the solutions of the initial value problem and the boundary value problem for impulsive functional - differential equations are given.

In chapter 4 the method of quasilinearization is applied to various problems of impulsive differential equations. The development of this method begins with paper [20]. Recently many authors applied the method of quasilinearization for finding approximate solutions of problems for first and second order ordinary differential equations ([34], [91], [101], [112]). Several results for scalar impulsive differential equations ([38], [42], [43], [118]) have also been obtained. The main advantage of the method of quasilinearization is the rapid convergence of the sequences of successive approroximations. In the book the quasilinearization is applied to the initial value problem and to boundary value problems for impulsive differential equations in the scalar and in the n-dimensional case.

Each chapter concludes with a section devoted to notes and bibliographical remarks.

The following book is addressed to a wide audience including professionals such as mathematicians, applied researchers and practitioners.

Introduction

There are many real life processes and phenomena that are characterized by rapid changes in their state. The duration of these changes is relatively short compared to the overall duration of the whole process and the changes turn out to be irrelevant to the development of the studied process. The mathematical models in such cases can be adequately created with the help of impulsive equations. Some examples of such processes can be found in Physics, Biology, population dynamics, ecology, pharmacokinetics, and others.

In the general case, the impulsive equations consist of two parts:

– differential equation, that defines the continuous part of the solution;

– impulsive part, that defines the instantaneous changes and the discontinuity of the solution.

The first part of the impulsive equations, that is described by differential equations, could consist of ordinary differential equations, integro-differential equations, functional-differential equations, partial differential equations, etc.

The second part of the impulsive equations is called a *jump condition*. The points, at which the impulses occur, are called *moments of impulses*. The functions, that define the amount of impulses, are called *impulsive functions*.

The type of the moments of impulses defines different types of impulsive equations. The two types of impulsive equations, considered in the book are:

– impulsive equations with fixed moments of impulses (the impulses occur at initially given fixed points);

– impulsive equations with variable moments of impulses (the impulses occur on initially given sets, i.e. the impulse occurs when the integral curve of the solution hits a given set).

Bellow we will give a brief description of the basic types of impulsive equations studied in this book.

A. Impulsive Differential Equations

First type. *Impulsive differential equations with fixed moments of impulses.*

Let the points $t_k \in \mathbf{R}$ be fixed such that $t_{k+1} > t_k$, $k = 0, 1, 2, \ldots$ and $\lim_{k\to\infty} t_k = \infty$.

Then the impulsive differential equations are expressed by the following two equalities:

differential equation $($*continuous part*$)$

$$x' = f(t,x) \qquad \text{for} \quad t \geq t_0 \;\; t \neq t_k, \tag{1}$$

impulsive part $($*jump condition*$)$

$$x(t_k + 0) - x(t_k - 0) = I_k(x(t_k - 0)), \qquad \text{for} \quad k = 1, 2, \ldots, \tag{2}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$,$I_k : \mathbf{R}^n \to \mathbf{R}^n$, $k = 1, 2, 3, \ldots$.

The initial value problem for the system of impulsive differential equations is defined by the equations (1), (2) and the initial condition

$$x(t_0) = x_0. \tag{3}$$

The solution of the initial value problem for the system of impulsive differential equations (1), (2), (3) is denoted by $x(t;t_0,x_0)$, and the maximal interval of the type $[t_0, \beta)$ on which the solution $x(t;t_0,x_0)$ is defined, we denote by $J(t_0,x_0)$.

We will describe the motion of the point (t,x) of the integral curve of the solution of the system of impulsive differential equations (1), (2) with initial condition (3) .

The point (t,x) starts its motion from the point (t_0,x_0) of the set $D \subset \mathbf{R} \times \mathbf{R}^n$ and continues to move along the integral curve $(t,x(t))$ of the solution of the corresponding ordinary differential equation (1) with initial condition (3) until moment $t_1 > t_0$, at which the point instantaneously moves from position (t_1,x_1) to position (t_1,x_1^+), where $x_1 = x(t_1)$, $x_1^+ = x_1 + I_1(x_1)$. Then the point continues its motion on the integral curve of the solution of the corresponding ordinary differential equation (1) with initial condition $x(t_1) = x_1^+$ until moment $t_2 > t_1$ at which it jumps and the amount of the jump is determined by the equality (2) and so on. Points t_k are the moments of impulses, and functions $I_k(x)$ are impulsive functions.

The solution of the impulsive differential equations could be:

– *piecewise continuous function*: at the moments of impulses at least one inequality $I_k(x) \not\equiv 0$ holds. Then we define $x(t_k) = \lim_{t\to t_k-0} x(t)$ and the solution has a point of discontinuity at point $t_k \in \mathbf{R}$;

– *continuous function*: for all natural numbers k equalities $I_k(x) = 0$ hold.

Furthermore we will assume that $x(t_k) = x(t_k - 0) = \lim_{t\to t_k-0} x(t) < \infty$ and $x(t_k + 0) = \lim_{t\to t_k+0} x(t) < \infty$.

We will give some examples to illustrate the behavior of the solutions of the impulsive differential equations with fixed moments of impulses.

Example 1. Consider the scalar impulsive differential equation

$$x' = 0, \qquad t \neq k, \;\; k = 1, 2, \ldots, \tag{4}$$

$$x(k+0) - x(k) = b. \tag{5}$$

The solution of the corresponding differential equation without impulses (4) with initial condition $x(0) = x_0$ is $x(t) = x_0$ for $t \geq 0$.

The solution of the impulsive differential equation (4), (5) with initial condition $x(0) = x_0$ is $x(t) = x_0 + kb$ for $t \in (k, k+1]$, $k = 1, 2, \ldots$. The solution is piecewise continuous function, which is increasing for $b > 0$, decreasing for $b < 0$, and for $b = 0$ the solution is equal to a constant and it coincides with the solution of the corresponding differential equation without impulses (4).

Example 2. Consider the scalar impulsive differential equation

$$x' = 0, \qquad t \neq k, \ \ k = 1, 2, \ldots \tag{6}$$

$$x(k+0) - x(k) = \frac{1}{x(k) - 1}, \ \ k = 1, 2, \ldots. \tag{7}$$

The solution of the corresponding differential equation without impulses (6) with initial condition $x(0) = 1$ is $x(t) = 1$ for $t \geq 0$.

The solution of the impulsive differential equation (6), (7) with initial condition $x(0) = 1$ is $x(t) = 1$ and it is defined only for $t \in [0, 1]$. The solution doesn't exist for $t > 1$, since the impulsive function $I(x) = \frac{1}{x-1}$ is undefined for $x = 1$.

Example 3. Consider the scalar impulsive differential equation

$$x' = 1 + x^2, \qquad t \neq \frac{\pi k}{4}, \ \ k = 1, 2, \ldots \tag{8}$$

$$x(k+0) - x(k) = -1, \quad t = \frac{\pi k}{4}. \tag{9}$$

with initial condition

$$x(0) = 0. \tag{10}$$

The solution of the corresponding differential equation without impulses (8) with initial condition (10) is $x(t) = tan\ t$. The solution is defined only on the interval $[0, \frac{\pi}{2})$ since $\lim_{t \to \frac{\pi}{2} - 0} tan\, t = \infty$.

The solution of the initial value problem for the impulsive differential equation (8), (9), (10) is

$$x(t) = \begin{cases} tan\, t, & \text{for } t \in [0, \frac{\pi}{4}] \\ tan(t - \frac{\pi}{4}), & \text{for } t \in (\frac{\pi(k)}{4}, \frac{\pi(k+1)}{4}], \ \ k = 1, 2 \ldots \end{cases}.$$

The solution is defined for $t \geq 0$ and it is a periodic function with a period $\frac{\pi}{4}$.

The examples above show that the solutions of the impulsive equations and the solutions of the corresponding ordinary differential equations without impulses have different behavior. This proves the necessity to independent studying the properties of the solutions of the impulsive equations.

Second type. *Impulsive differential equations with variable moments of impulses.*

Let the sequence of sets $\sigma_k = \{(t, x) \in \mathbf{R} \times \mathbf{R}^n : \ t = \tau_k(x)\}, \ \ k = 1, 2, \ldots$ be given, where the functions $\tau_k(x)$ are such that $\tau_k(x) < \tau_{k+1}(x)$, and the sequence $\{\tau_k(x)\}_{k=1}^{\infty}$ coverges uniformly in $x \in \mathbf{R}^n$ to ∞.

In this case the impulsive differential equations are composed of the following two equalities:

differential equations(*continuous part*)

$$x' = f(t,x) \quad \text{for } t \neq \tau_k(x),\ k = 1,2,\dots, \tag{11}$$

impulsive part (*jump condition*)

$$x(t+0) - x(t-0) = I_k(x(t)) \quad \text{for } t = \tau_k(x(t)), \quad k = 1,2,\dots, \tag{12}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $k = 1,2,3,\dots$.

We will give a description of the motion of the point (t,x) from the integral curve of the solution of impulsive differential equations (11), (12).

Point (t,x) starts its motion from point (t_0,x_0) in set $D \subset \mathbf{R} \times \mathbf{R}^n$ and continues to move along the integral curve $(t,x(t))$ of the corresponding ordinary differential equation (11) with initial condition $x(t_0) = x_0$ until moment $t_1 > t_0$, at which the integral curve meets set σ_{k_1}, i.e. until moment $t_1 > t_0$, at which the equality $t_1 = \tau_{k_1}(x(t_1))$ holds. At that moment the point moves instantaneously from position (t_1,x_1) to position (t_1,x_1^+), where $x_1 = x(t_1)$, $x_1^+ = x_1 + I_{k_1}(x_1)$ and it continues to move on the integral curve of the solution of the corresponding ordinary differential equation (11) with initial condition $x(t_1) = x_1^+$ until moment $t_2 > t_1$, at which the integral curve meets set σ_{k_2}, then point jumps instantaneously, and the amount of the jump is defined by equality (12) and so on.

We will note that in the case of impulses on given sets the moments of impulses are unknown initially. The moments of impulses depend not only on the right part of the equation, but also on the initial condition and on the functions $\tau_k(x)$, that define the sets of impulses. For example, if the initial value problem for the impulsive equations has two solutions, these solutions can have different points of discontinuity.

The solutions of impulsive equations with variable moments of impulses have unique properties. One of these properties is called *beating*. This is the case when the integral curve of the solution intersects the same set more than once. This could be the reason for nonexistence of the solution over the whole given interval.

We will give examples to illustrate some typical properties of the solutions of the impulsive differential equations with variable moments of impulses.

Example 4. Consider the initial value problem for the linear impulsive differential equation

$$x' = 0, \qquad t \neq \tau(x), \tag{13}$$

$$x(t+0) - x(t) = x(t), \quad t = \tau(x), \tag{14}$$

$$x(0) = 1, \tag{15}$$

where $x \in \mathbf{R}$, $\tau(x) = arctan\, x$.

The solution of the initial value problem for the impulsive equation (13), (14), (15) is

$$x(t;0,1) = i \quad \text{for } arctan(i-1) < t \leq arctan\, i,\ i = 1,2,\dots.$$

The solution is defined only on the interval $[0,\frac{\pi}{2})$, since $\lim_{i\to\infty} arctan\ i = \frac{\pi}{2}$. The integral curve of the solution of the initial value problem for the impulsive equation (13), (14), (15) intersects infinitely many times the curve $\sigma = \{(t,x) \in \mathbf{R} \times \mathbf{R} : \ t = arctan\, x\}$ at moments $\tau_i = arctan\, i, \ i = 1,2,\ldots$, therefore there is a *beating*.

Example 5 ([89]). Consider the scalar impulsive differential equation

$$x' = 0, \qquad t \neq \tau_k(x), \ \ k = 0,1,2,\ldots, \tag{16}$$

$$x(t+0) - x(t) = I_k(x), \quad t = \tau_k(x), \tag{17}$$

where $x \in \mathbf{R}$, $\tau_k(x) = x + 6k$, $I_k(x) = x^2 signx - x, \ k = 0,1,2,\ldots$.

The solution of the impulsive differential equation (16), (17) with initial condition $x(0) = x_0$ for $|x_0| \geq 3$ has no impulses, since the integral curve of the solution does not intersect any set $\sigma_k = \{(t,x) \in \mathbf{R} \times \mathbf{R} : \ t = x + 6k\}$, $k = 1,2,\ldots$. Therefore the solution is a continuous function, defined on $[0,\infty)$.

For $x_0 \in (1,3)$ the solution of the impulsive differential equation (16), (17) with initial condition $x(0) = x_0$ has finite number of impulses. For example, for $x_0 = \sqrt[4]{2}$ the solution of the impulsive differential equation (16), (17) with the initial condition $x(0) = x_0$ has three moments of impulses at points, when the integral curve of the solution intersects set $\sigma_0 = \{(t,x) \in \mathbf{R} \times \mathbf{R} : \ t = x\}$. For $t > t_3 = 2$ the integral curve of the solution meets no set σ_k. The solution is a piecewise continuous function, defined on $[0,\infty)$.

For $x_0 \in (0,1)$ the solution of the impulsive differential equation (16), (17) with initial condition $x(0) = x_0$ has the property *beating*, i.e. the integral curve of the solution intersects sets $\sigma_k = \{(t,x) \in \mathbf{R} \times \mathbf{R} : \ t = x + 6k\}$ at infinite number of moments t_k, for which $t_k = x(t_k) + 6k$, $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = 0$.

For $x_0 \in (-1,0)$ the solution of the impulsive differential equation (16), (17) with initial condition $x(0) = x_0$ has the property *beating*, i.e. the integral curve of the solution intersects set $\sigma_k = \{(t,x) \in \mathbf{R} \times \mathbf{R} : \ t = x + 6k\}$ at infinite number of points t_k, for which the equality $t_k = x(t_k) + 6k$ holds, and $\lim_{k\to\infty} t_k = 6$, and $\lim_{k\to\infty} x(t_k) = 0$.

For $x(0) = 0$, $x(0) = 1$ or $x(0) = -1$ the integral curve of the solution of the impulsive differential equation (16), (17) with initial condition $x(0) = x_0$ crosses sets σ_k at infinite number of points t_k, but $I_k(x(t_k)) = x^2(t_k) sign\, x(t_k) - x(t_k) = 0$, so the solution has no impulses and it is a continuous function at points t_k.

We note that sufficient conditions for absence of the phenomenon *beating* for impulsive ordinary differential equations are given in ([41], [89]).

B. Impulsive Differential-Difference Equations

In many real life processes and phenomena the dynamics of the studied system at the present moment depends on the behavior of the system in some previous moments. For modeling the dynamics of such processes it makes sense to use functional differential equations, a partial case of which are differential-difference equations. One of the initial works in the theory of functional-differential equations is monograph of Volterra [119]. In the last decades different qualitative properties of the solutions of different types of problems have been obtained (see monographs [61], [62], [85], [86] and cited therein references).

Some real processes are characterized that their dynamics at present depend not only on the behavior of the processes at some previous moments but also at some moments the processes change their behavior instantaneously. In this case it makes sense to use impulsive differential-difference equations as models of such processes.

We will describe the impulsive differential-difference equations and their solutions, that are object of investigation in the current book.

First type. *Impulsive differential-difference equations with fixed moments of impulses.*

Let the points $t_k \in \mathbf{R}$ be fixed such that $t_{k+1} > t_k$, $k = 0,1,2,\dots$ and $\lim_{k\to\infty} t_k = \infty$.

Then the impulsive differential-difference equations can be written with the help of both equations:

differential-difference equation (*continuous part*)

$$x' = f(t, x(t), x(t-h)) \quad \text{for} \quad t \geq t_0, \quad t \neq t_k, \quad k = 1,2,\dots, \tag{18}$$

impulsive part (*jump condition*)

$$x(t_k + 0) - x(t_k - 0) = I_k(x(t_k)) \qquad \text{for} \quad k = 1,2,\dots, \tag{19}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $(k = 1,2,3,\dots)$, and $h = const > 0$.

Consider the initial value problem for the system of impulsive differential - difference equations, that is defined by the equations (18), (19) and initial condition

$$x(t) = \varphi(t) \quad \text{for} \;\; t \in [t_0 - h, t_0], \tag{20}$$

where $\varphi : [t_0 - h, t_0] \to \mathbf{R}^n$.

We denote the solution of the initial value problem for the system of impulsive differential-difference equations (18), (19), (20) by $x(t;t_0,\varphi)$, and we denote by $J(t_0,\varphi)$ the maximal interval of the type $[t_0 - h, \beta)$ on which the solution $x(t;t_0,\varphi)$ is defined.

We will give a short description of the solution $x(t;t_0,\varphi)$ of the initial value problem for the system of impulsive differential-difference equations (18), (19), (20).

For $t \in [t_0 - h, t_1]$ the solution $x(t;t_0,\varphi)$ coincides with the solution $X_1(t;\varphi)$ of the initial value problem for the corresponding system of differential-difference equations without impulses

$$x' = f(t, x(t), x(t-h)) \quad \text{for} \quad t \geq t_0,$$
$$x(t) = \varphi(t) \quad \text{for} \;\; t \in [t_0 - h, t_0].$$

For $t \in (t_1, t_2]$ the solution $x(t;t_0,x_0)$ coincides with the solution $X_2(t;\varphi_1)$ of the initial value problem for the corresponding system of differential-difference equations without impulses

$$x' = f(t, x(t), x(t-h)) \quad \text{for} \quad t \geq t_1,$$
$$x(t) = \varphi_1(t) \qquad\qquad \text{for} \;\; t \in [t_1 - h, t_1],$$

where

$$\varphi_1(t) = \begin{cases} X_1(t;\varphi) & \text{for} \;\; t \in [t_1 - h, t_1), \\ X_1(t_1;\varphi) + I_1(X_1(t_1;\varphi)) & \text{for} \;\; t = t_1, \end{cases}$$

and so on.

Therefore the function $x(t;t_0,x_0)$ is a piecewise continuous function on $J(t_0,x_0)$.

Example 6. Consider the initial value problem for the linear scalar impulsive differential-difference equation

$$x' = x(t-2) \quad \text{for} \quad t \geq 0, \quad t \neq k, \ \ k = 1,2,\ldots, \tag{21}$$

$$x(k+0) = b(x(k)) \quad \text{for} \ \ k = 1,2,\ldots, \tag{22}$$

$$x(t) = a \quad \text{for } t \in [-2,0], \tag{23}$$

where $x \in \mathbf{R}$, a, b are constants.

The solution of the initial value problem (21), (22), (23) exists on the interval $[-2,\infty)$ and is defined by

$$x(t) = \begin{cases} a, & \text{for } \ t \in [-2,0], \\ a(t+1), & \text{for } \ t \in (0,1], \\ 2ab + a(t-1), & \text{for } \ t \in (1,2], \\ ab(2b+1) + \frac{a}{2}(t-2)(t+4), & \text{for } \ t \in (2,3], \\ ab[b(2b+1)+\frac{7}{2}] + \frac{a}{2}(t-3)(t+4b+1), & \text{for } \ t \in (3,4], \\ \cdots\cdots\cdots \end{cases}$$

The solution is a piecewise continuous function with points of discontinuity at $k = 1,2,\ldots.$

For $b = 1$ the solution of the initial value problem (21), (22), (23) is defined by

$$x(t) = \begin{cases} a, & \text{for } \ t \in [-2,0], \\ a(t+1), & \text{for } \ t \in (0,2], \\ \frac{a}{2}(t^2+2t-2), & \text{for } \ t \in (2,4] \cdots\cdots\cdots \end{cases}$$

and coincides with the solution of the corresponding differential-difference equation

$$x' = x(t-2) \quad \text{for} \quad t \geq 0$$

with initial condition

$$x(t) = a, \quad \text{for } t \in [-2,0],$$

that is defined on $[-2,\infty)$.

Second type. *Impulsive differential-difference equations with variable moments of impulses.*

In the case when the impulses occur on given sets, the description of the solution is similar to that in the case of fixed moments of impulses. In this case there are some unique phenomena. One of these phenomena is so called *beating*, i.e. when the integral curve of the solution meets one and the same set more than once (finite or infinity many times).

C. Impulsive Hybrid Equations

Most modern engineering systems are highly interconnected and interdependent on the various parts of the information and communication networks that comprise them. This complexity imposes the use of impulsive hybrid systems for the purpose of mathematical modeling. Impulsive hybrid systems are generally used when the modeled system is comprised of analog components, described by differential equations, and digital components which have the ability to change the state of the system in an instant. These real world systems also contain a conditional control algorithm for the digital component. The change of state in the modeled system has been described in [5], [59], [103], and [111]. Impulsive differential equations are also a natural fit in Biology, Chemistry, Physics, Pharmacology, and Medicine. For example, the human blood pressure is continuously regulated. However at times of stress, certain organs, such as the heart and the brain, get priority over the other organs. Another example comes from Genetics, where we also have a combination of discrete components and continuous components. So far, such systems have been modeled as discrete systems which decision greatly limits the possibility for investigation of the components of such systems. The development of impulsive hybrid equations allows for a more adequate investigation of complex real life systems such as the ones described above.

First type. *Impulsive hybrid equations with fixed points of impulses.*

Let points $\{t_k\}_0^\infty$ be fixed such that $t_{k+1} > t_k$, $k = 0,1,2,\dots$ and $\lim_{k\to\infty} t_k = \infty$.

Then the impulsive hybrid equations can be written by the help with the following two equations:

hybrid differential equation (*continuous part*)

$$x' = f(t, x(t), \lambda_k(x(t_k))) \quad \text{for} \quad t \ge t_0, \quad t \ne t_k, \quad k = 1,2,\dots, \tag{24}$$

impulsive part (*jump condition*)

$$x(t_k + 0) = x(t_k - 0) + I_k(x(t_k)) \qquad \text{for} \quad k = 1,2,\dots, \tag{25}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $\lambda_k : \mathbf{R}^n \to \mathbf{R}^m$, $k = 1,2,3,\dots$.

Consider the initial value problem for the impulsive hybrid equations, defined by equations (24), (25) with initial condition

$$x(t_0) = x_0. \tag{26}$$

We denote the solution of the initial value problem for the system of impulsive hybrid equations (24), (25), (26) by $x(t;t_0,x_0)$, and we denote by $J(t_0,x_0)$ the maximal interval of the type $[t_0,\beta)$, on which the solution $x(t;t_0,x_0)$ is defined.

We will give a brief description of the solution $x(t;t_0,x_0)$ of the initial value problem for the system of impulsive hybrid equations (24), (25), (25).

For $t \in [t_0,t_1]$ the solution $x(t;t_0,x_0)$ coincides with the solution $X_1(t;t_0)$ of the initial value problem for the ordinary differential equation without impulses

$$x' = f(t, x(t), \lambda_0(x_0)) \quad \text{for} \quad t \ge t_0,$$

$$x(t_0) = x_0.$$

For $t \in (t_1,t_2]$ the solution $x(t;t_0,x_0)$ coincides with solution $X_2(t;t_1)$ of the initial value problem for the ordinary differential equation without impulses

$$x' = f(t, x(t), \lambda_1(X_1(t_1;t_0)) \quad \text{for} \quad t \ge t_1,$$

$$x(t_1) = X_1(t_1;t_0) + I_1(X_1(t_1;t_0)),$$

and so on.

Therefore function $x(t;t_0,x_0)$ is a piecewise continuous on $J(t_0,x_0)$.

D. Impulsive Differential Equations with "Supremum"

Differential equations with "supremum" are adequate mathematical models of various real world processes. They find application, for example, in the theory of automatic regulation ([109]). As a simple example of mathematical simulation by means of such equations we will consider the system for regulating the voltage of a generator of constant current. The object of experiment is a generator of constant current with parallel simulation and regulated quantity is the voltage at the source electric current. The equation describing the work of the regulator has form ([109])

$$T_0 u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t),$$

where T_0 and q are constants characterizing the object, $u(t)$ is the voltage regulated and $f(t)$ is the perturbed effect.

In the case when the equation consists only of differential equation, and there are no impulses, the equation is called equation with $''$maximum$''$. Some properties of the solutions of differential equations with $''$maximum$''$ are studied in [4], [12], [108], [121].

In the case when a jump condition is added to the differential equation, the equation is called an impulsive differential equation with "supremum".

First type. *Impulsive differential equations with "supremum" and fixed moments of impulses.*

In the common case the impulsive differential equations with "supremum" are expressed with the following two equations:

differential equation with "supremum" (*continuous part*)

$$x' = f(t, x(t), \sup_{s\in[t-h,t]} x(s)) \quad \text{for} \quad t \geq t_0, \quad t \neq t_k, \quad k = 1,2,\ldots, \tag{27}$$

impulsive part (*jump condition*)

$$x(t_k+0) = x(t_k-0) + I_k(x(t_k)) \qquad \text{for} \quad k = 1,2,\ldots, \tag{28}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $k = 1,2,3,\ldots$, $h = const > 0$.

Consider the initial value problem for the impulsive differential equations with "supremum", that is defined by equations (27), (28) and initial condition

$$x(t+t_0) = \varphi(t) \quad \text{for} \quad t \in [-h,0], \tag{29}$$

where $\varphi : [-h,0] \to \mathbf{R}^n$.

We denote the solution of the initial value problem (27), (28), (29) by $x(t;t_0,\varphi)$ and by $J(t_0,\varphi)$ – the maximal interval in which $x(t;t_0,\varphi)$ is defined.

We will give a description of solution $x(t;t_0,\varphi)$ of the initial value problem (27), (28), (29):

(a) For $t\in[t_0-h,t_0]$ the solution $x(t;t_0,\varphi)$ coincides with the function $\varphi(t-t_0)$.

(b) For $t\in[t_0,t_1]$ the solution $x(t;t_0,\varphi)$ coincides with solution $X_1(t;t_0)$ of the differential equations with "supremum" without impulses

$$x' = f(t,x(t),\sup_{s\in[t-h,t]} x(s))$$

with initial condition

$$x(t+t_0)=\varphi(t) \ \text{ for } t\in[-h,0].$$

For $t\in(t_1,t_2]$ solution $x(t;t_0,\varphi)$ coincides with solution $X_2(t;t_1)$ of the initial value problem

$$x' = f(t,x(t),\sup_{s\in[t-h,t]} x(s))$$

$$x(t+t_1)=\varphi_1(t) \ \text{ for } \ t\in[-h,0],$$

where

$$\varphi_1(t)=\begin{cases} X_1(t+t_1;t_0) & \text{for } \ t\in[-h,0), \\ X_1(t_1;t_0)+I_1(X_1(t_1;t_0)) & \text{for } \ t=0, \end{cases}$$

and so on.

Therefore function $x(t;t_0,\varphi)$ is piecewise continuous in $J(t_0,\varphi)$.

Second type. *Impulsive differential equations with "supremum" and variable moments of impulses.*

In the case when the impulses occur on initially given hypersurfaces the solutions of the impulsive differential equations with "supremum" and variable moments of impulses are defined in analogous way as in the case of fixed moments of impulses.

Next we will present the basic definitions used in the book.

Definition 1. The set $PC(\Omega,\mathbf{R}^n)$, where $\Omega\subset\mathbf{R}$, is called the set of all functions $u:\Omega\to\mathbf{R}^n$, that are piecewise continuous in Ω, i.e. there exist limits $\lim_{t\downarrow t_k}u(t)=u(t_k+0)<\infty$ and $\lim_{t\uparrow t_k}u(t)=u(t_k-0)=u(t_k)<\infty, t_k\in\Omega$.

Definition 2. The set $PC^1(\Omega,\mathbf{R}^n)$ is called the set of all functions $u\in PC(\Omega,\mathbf{R}^n)$, that are continuously differentiable for all $t\in\Omega$ in which the function is continuous and there exist left derivatives at the points of discontinuity.

Let the sequence of functions $\tau_k\in C(\mathbf{R}^n,\mathbf{R})$, $k=1,2,\ldots$ be given such that $\tau_k(x)<\tau_{k+1}(x)$ for $x\in\mathbf{R}^n$ and the sequence $\{\tau_k(x)\}_{k=1}^{\infty}$ coverges uniformly in $x\in\mathbf{R}^n$ to ∞.

Sets σ_k,G_k,D_k, $k=1,2,\ldots$ are defined by the equalities

$$\sigma_k=\{(t,x)\in\mathbf{R}\times\mathbf{R}^n:\ t=\tau_k(x)\}, \tag{30}$$

$$G_k=\{(t,x)\in[0,\infty)\times\mathbf{R}^n:\ \tau_{k-1}(x)<t<\tau_k(x)\}, \tag{31}$$

$$D_k = \{(t,x) \in [0,\infty) \times \mathbf{R}^n : \ \tau_{k-1}(x) < t \le \tau_k(x)\}. \tag{32}$$

Note that the sets defined by (30), (31), (32) are related to the impulsive equations with variable moments of impulses. In the case of fixed moments of impulses $t_k, (k = 1, 2, \ldots)$, sets σ_k, G_k, D_k, $k = 1, 2, \ldots$ are reduced to sets

$$\sigma_k = \{(t_k, x) : \ \ x \in \mathbf{R}^n\},$$

$$G_k = \{(t,x) : \ \ x \in \mathbf{R}^n, \ t \in (t_{k-1}, t_k)\},$$

and

$$D_k = \{(t,x) : \ x \in \mathbf{R}^n, \ t \in (t_{k-1}, t_k]\}.$$

Chapter 1

Method of the Integral Inequalities for Qualitative Investigations of Impulsive Equations

Integral inequalities are a powerful mathematical tool for investigating qualitative characteristics of the solutions of differential equations. The current book deals with the generalizations of the classical Gronwall-Bellman and Bihari integral inequalities and the corresponding proofs of the analogues of the before mentioned inequalities for piecewise continuous functions. Some of the obtained results are used to investigate the properties of the solutions of concrete problems for different types of impulsive equations.

It is also worth mentioning that an overview of the theory of integral inequalities for discontinuous functions is presented in the following monographs [17], [45] and [90] and some linear inequalities for piecewise continuous functions, similar to the well known Gronwall-Bellman inequality, are proved in [15], [89].

We note similar results to those obtained in this chapter are published in [11], [14], [66], [107].

Remark 1. In the chapter 1 the points $t_k \geq 0, k = 1,2,\dots$ are fixed such that $t_k \leq t_{k+1}$ and $\lim_{k\to\infty} t_k = \infty$.

1.1. Linear Impulsive Integral Inequalities

Several linear integral inequalities of Gronwall type for piecewise continuous scalar functions will be given in this section. The importance of these type of inequalities is defined by their applications for various types of qualitative investigations and their applications in proofs of inequalities of other types.

We will begin with a generalization of the classical Gronwall–Bellman inequality for continuous functions, that will be used in the further proofs.

Lemma 1.1.1 ([40]). *Let the following conditions be satisfied:*

1. *Function* $f(t) \in C([0,\infty),[0,\infty))$.
2. *Function* $n(t) \in C([0,\infty),[0,\infty))$ *is nondecreasing.*

3. *Function* $g(t) \in C([0,\infty),[1,\infty))$.
4. *Function* $u(t) \in C([0,\infty),[0,\infty))$ *satisfies the inequality*

$$u(t) \leq n(t) + g(t) \int_0^t f(s)u(s)ds.$$

Then for $t \geq 0$ *the inequality*

$$u(t) \leq n(t)g(t)e^{\int_0^t f(s)g(s)ds}$$

holds.

We will prove a linear impulsive integral inequality.

Theorem 1.1.1. *Let the following conditions be satisfied:*
1. *Function* $v(t) \in PC([0,\infty),[0,\infty))$.
2. *Function* $p(t) \in PC([0,\infty),[1,\infty))$ *is nondecreasing.*
3. *Function* $u(t) \in PC([0,\infty),[0,\infty))$ *satisfies the inequality*

$$u(t) \leq p(t)\big[c + \sum_{0<t_i<t} \beta_i u(t_i) + \int_0^t v(s)u(s)ds\big], \tag{1.1}$$

where $c \geq 0,\ \beta_i \geq 0,\ (i = 1,2,\dots)$ *are constants.*
Then for $t \geq 0$ *the inequality*

$$u(t) \leq cp^2(t) \prod_{0<t_i<t} p(t_i)(1+\beta_i p(t_i))e^{\int_0^t v(s)p(s)ds} \tag{1.2}$$

holds.

Proof. Let $t \in [0,t_1]$. Inequality (1.1) can be rewritten in the form

$$u(t) \leq p(t)\big[c + \int_0^t v(s)u(s)ds\big].$$

According to Lemma 1.1.1 for $t \in [0,t_1]$ the inequality

$$u(t) \leq cp^2(t)e^{\int_0^t v(s)p(s)ds} \tag{1.3}$$

holds. Therefore Theorem 1.1.1 stands for $t \in [0,t_1]$.

We will use mathematical induction to prove the proposition.

Assume that Theorem 1.1.1 is true for a natural number $k > 1$, i.e. inequality (1.2) holds for $t \in [0,t_k]$.

Let $t \in (t_k,t_{k+1}]$. From inequality (1.1) follows the validity of

$$\begin{aligned} u(t) \quad \leq \quad & p(t)\big[c + \sum_{i=1}^k \beta_i u(t_i) + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} v(s)u(s)ds\big] \\ & + p(t)\int_{t_k}^t v(s)u(s)ds. \end{aligned} \tag{1.4}$$

Denote

$$g(t) = p(t)\big[c + \sum_{i=1}^{k} \beta_i u(t_i) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} v(s)u(s)ds\big].$$

According to the inductive assumption we obtain the following bound for function $g(t)$

$$\begin{aligned} g(t) \quad &\leq \quad p(t)\big[c + \sum_{i=1}^{k} \beta_i c p^2(t_i) \prod_{j=1}^{i-1} p(t_j)(1+\beta_j p(t_j)) e^{\int_0^{t_i} v(s)p(s)ds} \\ &+ \quad \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} v(s) c p^2(s) \prod_{0<t_j<s} p(t_j)(1+\beta_j p(t_j)) e^{\int_0^t v(\tau)p(\tau)d\tau} ds\big]. \end{aligned}$$

Since function $p(t)$ is nondecreasing, the inequality

$$\begin{aligned} g(t) \quad &\leq \quad cp(t)\big[1 + \sum_{i=1}^{k} \beta_i p^2(t_i) \prod_{j=1}^{i-1} p(t_j)(1+\beta_j p(t_j)) e^{\int_0^{t_i} v(s)p(s)ds} \\ &+ \quad \sum_{i=1}^{k} p(t_i) \prod_{j=1}^{i-1} p(t_j)(1+\beta_j p(t_j)) \int_{t_{i-1}}^{t_i} v(s)p(s) e^{\int_0^t v(\tau)p(\tau)d\tau} ds\big] \\ &\leq \quad cp(t)\Big\{1 + \sum_{i=1}^{k} \beta_i p^2(t_i) \prod_{j=1}^{i-1} p(t_j)(1+\beta_j p(t_j)) e^{\int_0^{t_i} v(s)p(s)ds} \\ &+ \quad \sum_{i=1}^{k} p(t_i) \prod_{j=1}^{i-1} p(t_j)(1+\beta_j p(t_j)) \Big[e^{\int_0^{t_i} v(s)p(s)ds} - e^{\int_0^{t_{i-1}} v(s)p(s)ds}\Big]\Big\} \end{aligned}$$

holds.

Therefore function $g(t)$ satisfies the inequality

$$g(t) \leq cp(t)\Big(\prod_{i=1}^{k} p(t_j)(1+\beta_j p(t_j))\Big) e^{\int_0^{t_k} v(s)p(s)ds}. \tag{1.5}$$

We substitute the upper bound (1.5) for function $g(t)$ into inequality (1.4) and apply Lemma 1.1.1 over $(t_k, t_{k+1}]$. We obtain that for $t \in (t_k, t_{k+1}]$ the inequaity

$$\begin{aligned} u(t) \quad &\leq \quad p(t)g(t) e^{\int_{t_k}^t v(s)p(s)ds} \\ &\leq \quad cp^2(t)\Big(\prod_{i=1}^{k} p(t_i)(1+\beta_i p(t_i))\Big) e^{\int_0^t v(s)p(s)ds} \end{aligned}$$

holds.

The last inequality proves Theorem 1.1.1. □

As a partial case of Theorem 1.1.1 we obtain the following result:

Corollary 1.1.1. *Let the conditions of Theorem* 1.1.1 *be satisfied for* $p(t) \equiv 1$, *and the function* $u(t)$ *satisfies the inequality*

$$u(t) \leq c + \sum_{0<t_i<t} \beta_i u(t_i) + \int_0^t v(s)u(s)ds, \tag{1.6}$$

where $c \geq 0,\ \beta_i \geq 0,\ (i = 1,2,\dots)$ *are constants.*

Then for $t \geq 0$ *the inequality*

$$u(t) \leq c\Big(\prod_{0<t_i<t}(1+\beta_i)\Big)e^{\int_0^t v(s)ds} \tag{1.7}$$

holds.

In the future we will use the following result:

Lemma 1.1.2 (Corollary 16.2, [17]). *Let the following conditions be satisfied:*

1. *Function* $b(t) \in C([0,\infty),[0,\infty))$.
2. *Function* $a(t) \in C([0,\infty),[0,\infty))$ *is nondecresing.*
3. *Function* $u(t) \in PC([0,\infty),[0,\infty))$ *and satisfies the inequality*

$$u(t) \leq a(t) + \sum_{0<t_i<t}\beta_i u(t_i) + \int_0^t b(s)u(s)ds, \tag{1.8}$$

where $\beta_i \geq 0,\ (i = 1,2,\dots)$ *are constants.*

Then for $t \geq 0$ *the inequality*

$$u(t) \leq a(t)\Big(\prod_{0<t_i<t}(1+\beta_i)\Big)e^{\int_0^t b(s)ds} \tag{1.9}$$

holds.

In the case when function $a(t)$ is differentiable we obtain better bound for the unknown function.

Theorem 1.1.2. *Let the following conditions be satisfied:*

1. *Function* $b(t) \in PC([0,\infty),[0,\infty))$.
2. *Function* $a(t) \in PC^1([0,\infty),[0,\infty))$ *and* $a'(t) \geq 0$ *for* $t \neq t_k$.
3. *Function* $u(t) \in PC([0,\infty),[0,\infty))$ *and satisfies the inequality*

$$u(t) \leq a(t) + \sum_{0<t_i<t}\beta_i u(t_i) + \int_0^t b(s)u(s)ds, \tag{1.10}$$

where $\beta_i \geq 0,\ (i = 1,2,\dots)$ *are constants.*

Then for $t \in (t_n, t_{n+1}],\ \ (n = 0,1,2,\dots,\ \ t_0 = 0)$ *the inequality*

$$\begin{aligned} u(t) \ &\leq \ a(0)e^{\int_0^t b(s)ds}\prod_{j=1}^{n}(1+\beta_j) \\ &+ \sum_{j=1}^{n}\Big\{\Big[\int_{t_{j-1}}^{t_j} e^{-\int_0^\tau b(s)ds}a'(\tau)d\tau\Big]\Big(\prod_{i=j}^{n}(1+\beta_i)\Big)e^{\int_0^t b(s)ds}\Big\} \\ &+ \int_{t_n}^{t} e^{-\int_0^\tau b(s)ds}a'(\tau)d\tau e^{\int_0^t b(s)ds} \end{aligned} \tag{1.11}$$

holds.

Proof. Let's define function $v(t)$ by the equality

$$v(t) = a(t) + \sum_{0<t_i<t} \beta_i u(t_i) + \int_0^t b(s)u(s)ds.$$

Function $v(t) \in PC^1([0,\infty),[0,\infty))$ satisfies the inequalities

$$\begin{aligned} &v' \le a'(t) + b(t)v(t), \quad t \ge 0, \ t \ne t_k, \\ &v(t_k+0) - v(t_k-0) \le \beta_k v(t_k), \ \ n = 1,2,\ldots, \end{aligned}$$

$$v(0) = a(0). \tag{1.12}$$

Consider the initial value problem for impulsive differential equation

$$\begin{aligned} &w' = a'(t) + b(t)w(t), \quad t \ge 0, \ t \ne t_k, \\ &w(t_k+0) = (1+\beta_k)w(t_k), \ \ n = 1,2,\ldots, \end{aligned}$$

$$w(0) = a(0). \tag{1.13}$$

According to the results for linear impulsive systems (see formula (2.52), [89]) the initial value problem (1.13) has a unique solution $w(t)$, that is given by the equality

$$\begin{aligned} w(t) &= a(0)e^{\int_0^t b(s)ds} \prod_{j=1}^{k} (1+\beta_j) \\ &+ \sum_{j=1}^{k} \Big\{ \Big[\int_{t_{j-1}}^{t_j} e^{-\int_0^\tau b(s)ds} a'(\tau)d\tau \Big] \Big(\prod_{i=j}^{k} (1+\beta_i) \Big) e^{\int_0^t b(s)ds} \Big\} \\ &+ \int_{t_k}^{t} e^{-\int_0^\tau b(s)ds} a'(\tau)d\tau e^{\int_0^t b(s)ds}, \quad t \in (t_k, t_{k+1}]. \end{aligned} \tag{1.14}$$

We will prove that inequality

$$v(t) \le w(t), \quad t \ge 0 \tag{1.15}$$

holds.

Define function $\varpi(t) = w(t) + \varepsilon e^{\alpha(t)}$ for $t \ge 0$, where $\varepsilon > 0$ is an arbitrary number, and function $\alpha(t) \in PC([0,\infty),[0,\infty))$ satisfies the inequalities

$$\alpha'(t) > b(t) \quad t \ge 0, \ t \ne t_i,$$

and

$$e^{\alpha(t_k+0)} > (1+\beta_k)e^{\alpha(t_k)}, \quad k = 1,2\ldots$$

Then $\varpi(t) \in PC([0,\infty),[0,\infty))$, $\varpi(0) = w(0) + \varepsilon e^{\alpha(0)} > a(0)$ and inequality

$$\begin{aligned} \varpi'(t) &= a'(t) + b(t)w(t) + \varepsilon e^{\alpha(t)}\alpha'(t) \\ &= a'(t) + b(t)\varpi(t) - b(t)\varepsilon e^{\alpha(t)} + \varepsilon e^{\alpha(t)}\alpha'(t) \end{aligned}$$

$$> a'(t) + b(t)\varpi(t), \quad t \ge 0, \ t \ne t_k \tag{1.16}$$

holds.

From the definition of function $\varpi(t)$ and the jump condition in (1.13), we obtain

$$\begin{aligned}\varpi(t_k+0) &= (1+\beta_k)w(t_k)+\varepsilon e^{\alpha(t_k+0)}\\ &= (1+\beta_k)\varpi(t_k)+\varepsilon e^{\alpha(t_k+0)}-(1+\beta_k)\varepsilon e^{\alpha(t_k)}\\ &> (1+\beta_k)\varpi(t_k),\ \ n=1,2,\dots.\end{aligned} \tag{1.17}$$

We will prove that

$$v(t)\le\varpi(t),\quad t\ge 0. \tag{1.18}$$

Assume the contrary, i.e. there exists a point $\tau>0$ such that

$$v(\tau)=\varpi(\tau), \tag{1.19}$$

and

$$v(t)<\varpi(t),\quad t\in(0,\tau). \tag{1.20}$$

Consider the following two cases:

Case 1. Let $\tau\ne t_k, k=1,2,\dots.$ Then inequality

$$v'(\tau)\ge\varpi'(\tau) \tag{1.21}$$

holds.

From inequalities (1.12), (1.16), (1.21), and equality (1.19) we obtain

$$0\ge\varpi'(\tau)-v'(\tau)>b(\tau)(\varpi(\tau)-v(\tau))=0.$$

The above contradiction proves the validity of (1.18) in this case.

Case 2. Let there exists a natural number k such that $\tau=t_k$. Then

$$v(t_k)\le\varpi(t_k),\quad v(t_k+0)=\varpi(t_k+0). \tag{1.22}$$

Similarly to the proof of case 1, from inequalities (1.12), (1.17) and relations (1.22) we obtain that

$$0=\varpi(t_k+0)-v(t_k+0)>(1+\beta_k)(\varpi(t_k)-v(t_k))\ge 0.$$

The above contradiction proves the validity of inequality (1.18) in this case.

Taking a limit into inequality (1.18) as $\varepsilon\to 0$ we obtain inequality (1.15). From inequality (1.15), equality (1.14), and the definition of the function $v(t)$ follows the validity of inequality (1.11). □

1.2. Nonlinear Impulsive Integral Inequalities

The simulations of most real world phenomena and processes require to use as models nonlinear equations. The nonlinearity in the equations involves applications of various types of nonlinear integral inequalities.

In this section several nonlinear integral inequalities for scalar piecewise continuous functions are proved. The considered inequalities are generalizations of the classical Bihari inequality.

From here on, we will say that the conditions (H) are satisfied if:

H1. Function $Q \in C([0,\infty),[0,\infty))$ is decreasing and $Q(u) > 0$ for $u > 0$.

H2. For $u \geq 0$ and $\alpha \geq 1$ the inequality $\frac{1}{\alpha}Q(u) \leq Q(\frac{u}{\alpha})$ holds.

H3. There exist numbers $M_k,\ k = 1,2,\dots$ such that for $x \geq y \geq 0$ inequality

$$G((1+\beta_k)x) - G((1+\beta_k)y) \leq M_k(G(x) - G(y)),$$

holds, where $G(u) = \int_{u_0}^{u} \frac{ds}{Q(s)},\ u_0 > 0$.

We will use the following nonlinear integral inequality of Bihari's type for continuous functions:

Lemma 1.2.1 ([39]). *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *Functions* $u(t), f(t), g(t) \in C([0,\infty),[0,\infty))$.
3. *There exists a constant* $a \geq 0$ *such that for* $t \geq 0$ *the inequality*

$$u(t) \leq a + \int_0^t f(s)u(s)ds + \int_0^t g(s)Q(u(s))ds$$

holds.

Then for $t \in [0,\gamma_1)$ *inequality*

$$u(t) \leq e^{\int_0^t f(s)ds} G^{-1}\Big\{G(a) + \int_0^t g(s)Q(e^{\int_0^s f(\tau)d\tau})ds\Big\},$$

holds, where G^{-1} *is the inverse function of function* $G(u)$,

$$\gamma_1 = \sup\big\{\xi \geq 0 : G(a) + \int_0^t g(s)Q(e^{\int_0^s f(\tau)d\tau})ds \in Dom(G^{-1}) fort \in [0,\xi]\big\}.$$

We will prove a generalization of Lemma 1.2.1 for piecewise continuous functions.

Theorem 1.2.1. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *Functions* $f(t), g(t) \in C([0,\infty),[0,\infty))$.
3. *Function* $u \in PC([0,\infty),[0,\infty))$ *and for* $t \geq 0$ *the inequality*

$$u(t) \leq a + \int_0^t f(s)u(s)ds + \int_0^t g(s)Q(u(s))ds + \sum_{0<t_k<t} \beta_k u(t_k), \tag{1.23}$$

holds, where $a = const > 0$, $\beta_k = const \geq 0,\ \ k = 1,2,\dots$.

Then for $t \in [0,\gamma_2)$ *the inequality*

$$u(t) \leq e^{\int_0^t f(s)ds} G^{-1}\Big\{G(R(t)) + \int_0^t g(s)\lambda(s,t)ds\Big\}, \tag{1.24}$$

holds, where $R(t) = a\prod_{0<t_k<t}(1+\beta_k)$ *for* $t \geq 0$, G^{-1} *is the inverse function of function* $G(u)$,

$$\lambda(s,t) = \Big(\prod_{s<t_j<t} M_j\Big) Q(e^{\int_0^s f(\tau)d\tau}),\ 0 < s \leq t, \tag{1.25}$$

$$\gamma_2 = \sup\big\{\xi \geq 0 : G(R(t)) + \int_0^t g(s)\lambda(s,t)ds \in Dom(G^{-1}) for\ t \in [0,\xi]\big\}.$$

Proof. We denote the right part of (1.23) by $z(t)$. Function $z \in PC([0,\infty),[0,\infty))$ is nondecreasing and satisfies both inequalities $u(t) \le z(t)$ and

$$z(t) \le a + \int_0^t f(s)z(s)ds + \int_0^t g(s)Q(z(s))ds + \sum_{0<t_k<t} \beta_k z(t_k). \tag{1.26}$$

We will prove that for $t \in [0,\gamma_2)$ function $z(t)$ satisfies the following inequality

$$z(t)e^{-\int_0^t f(s)ds} \le G^{-1}\Big\{G(R(t)) + \int_0^t g(s)\lambda(s,t)ds\Big\}. \tag{1.27}$$

Let $t \in [0,t_1]\bigcap[0,\gamma_2)$. Then from inequality (1.26) we obtain

$$z(t) \le a + \int_0^t f(s)z(s)ds + \int_0^t g(s)Q(z(s))ds. \tag{1.28}$$

According to Lemma 1.2.1 from inequality (1.28) we obtain

$$z(t)e^{-\int_0^t f(s)ds} \le G^{-1}\Big\{G(a) + \int_0^t g(s)Q(e^{\int_0^s f(\tau)d\tau})ds\Big\},$$

that proves the validity of (1.27) for $t \in [0,t_1]\bigcap[0,\gamma_2)$.

We will use mathematical induction to prove inequality (1.27). Assume that inequality (1.27) holds over $(t_{k-1},t_k]\bigcap[0,\gamma_2)$, where $k \ge 2$ is a natural number. Then the inequality

$$z(t_k)e^{-\int_0^{t_k} f(s)ds} \le G^{-1}\Big\{G(R(t_k)) + \int_0^{t_k} g(s)\lambda(s,t_k)ds\Big\} \tag{1.29}$$

holds.

Let $t \in (t_k,t_{k+1}]\bigcap[0,\gamma_2)$. The definition of function $z(t)$ and the properties of function $Q(u)$ imply that

$$\begin{aligned} z(t) &= a + \int_0^{t_k} f(s)u(s)ds + \int_0^{t_k} g(s)Q(u(s))ds + \sum_{i=1}^{k-1} \beta_i u(t_i) \\ &+ \beta_k u(t_k) + \int_{t_k}^t f(s)u(s)ds + \int_{t_k}^t g(s)Q(u(s))ds \\ &\le (1+\beta_k)z(t_k) + \int_{t_k}^t f(s)z(s)ds + \int_{t_k}^t g(s)Q(z(s))ds. \end{aligned} \tag{1.30}$$

Consider function $v \in C([t_k,t_{k+1}],[0,\infty))$, defined by the equalities

$$v(t) = \begin{cases} z(t)e^{-\int_0^{t_k} f(s)ds} & \text{for } t \in (t_k,t_{k+1}] \\ \Big(z(t_k) + \beta_k u(t_k)\Big)e^{-\int_0^{t_k} f(s)ds} & \text{for } t = t_k. \end{cases}$$

For $t \in (t_k,t_{k+1}]$ the inequality $v(t) \le z(t)$ holds. We multiply inequality (1.30) by the constant $e^{-\int_0^{t_k} f(s)ds}$ and we obtain that for $t \in (t_k,t_{k+1}]$ the inequality

$$v(t) \le (1+\beta_k)v(t_k) + \int_{t_k}^t f(s)v(s)ds + \int_{t_k}^t g(s)e^{-\int_0^{t_k} f(\xi)d\xi}Q(v(s))ds$$

$$\leq (1+\beta_k)v(t_k) + \int_{t_k}^{t} f(s)v(s)ds + \int_{t_k}^{t} g(s)Q(v(s))ds \tag{1.31}$$

holds.

Since $v(t_k) \leq (1+\beta_k)v(t_k)$, we conclude that inequality (1.31) holds for $t \in [t_k, t_{k+1}]$.

From inequality (1.31) and Lemma 1.2.1 we obtain

$$v(t)e^{-\int_{t_k}^{t} f(s)ds} \leq G^{-1}\Big\{G((1+\beta_k)v(t_k)) + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds\Big\} \tag{1.32}$$

for $t \in [t_k, t_{k+1}]$

We consider the following two cases:

Case 1. Let $v(t_k) \leq R(t_k) = a\prod_{j=1}^{k-1}(1+\beta_j)$. From the monotonicity of function $G(u)$ follows that

$$G\big((1+\beta_k)v(t_k)\big) \leq G\big((1+\beta_k)a\prod_{j=1}^{k-1}(1+\beta_j)\big) = G\big(a\prod_{j=1}^{k}(1+\beta_j)\big) = G(R(t)). \tag{1.33}$$

From inequalities (1.32) and (1.33) we obtain

$$\begin{aligned} G\Big(z(t)e^{-\int_0^t f(s)ds}\Big) &\leq \Big\{G(R(t)) + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds\Big\} \\ &\leq G(R(t)) + \int_{t_k}^{t} g(s)\lambda(s,t)ds. \end{aligned}$$

Case 2. Let $v(t_k) > R(t_k)$. From condition H3 we obtain the inequalities

$$\begin{aligned} &G\big((1+\beta_k)v(t_k)\big) - G\big(R(t_{k+1})\big) = G\big((1+\beta_k)v(t_k)\big) - G\big((1+\beta_k)R(t_{k+1})\big) \\ &\quad \leq M_k\big[G(v(t_k)) - G(R(t_k))\big]. \end{aligned}$$

From the definition of function $v(t)$ and inequality (1.32) we obtain

$$\begin{aligned} &G\Big(z(t)e^{-\int_0^t f(s)ds}\Big) = G\Big(v(t)e^{-\int_{t_k}^{t} f(s)ds}\Big) \\ &\leq G\Big((1+\beta_k)v(t_k)\Big) + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds \\ &\leq G(R(t_{k+1})) + M_k\Big(G(v(t_k)) - G(R(t_k))\Big) \\ &\qquad + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds. \end{aligned} \tag{1.34}$$

From inequality (1.29) follows the validity of

$$G(v(t_k)) \leq G(R(t_k)) + \int_0^{t_k} g(s)\lambda(s,t_k)ds. \tag{1.35}$$

Inequalities (1.34), (1.35) and the equality $R(t_{k+1}) = R(t)$ for $t \in (t_k, t_{k+1}]$ imply that

$$\begin{aligned} &G\Big(z(t)e^{-\int_0^t f(s)ds}\Big) \leq G(R(t_{k+1})) + M_k\Big(G(R(t_k)) - G(R(t_k)) \\ &\quad + \int_0^{t_k} g(s)\lambda(s,t)ds\Big) + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds \\ &= G(R(t_{k+1})) + \int_0^{t_k} g(s)M_k\lambda(s,t)ds + \int_{t_k}^{t} g(s)Q\big(e^{\int_{t_k}^{s} f(\xi)d\xi}\big)ds \\ &\leq G(R(t)) + \int_0^{t} g(s)\lambda(s,t)ds. \end{aligned} \tag{1.36}$$

From inequality (1.36) follows the validity of inequality (1.27) on the interval $t \in (t_k, t_{k+1}] \bigcap [0, \gamma_2)$. Therefore inequality (1.27) holds for all $t \in [0, \gamma_2)$.

From inequalities (1.27) and $u(t) \leq z(t)$ we obtain inequality (1.24). □

As a partial case of Theorem 1.2.1 we obtain the following result:

Corollary 1.2.2. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *Functions* $f(t), g(t) \in C([0,\infty),[0,\infty))$.
3. *Function* $a(t) \in C([0,\infty),[0,\infty))$ *is nondecreasing.*
4. *Function* $u \in PC([0,\infty),[0,\infty))$ *and for* $t \geq 0$ *the inequality*

$$u(t) \leq a(t) + \int_0^t f(s)u(s)ds + \int_0^t g(s)Q(u(s))ds + \sum_{0<t_k<t} \beta_k u(t_k) \tag{1.37}$$

holds.

Then for $t \in [0, \gamma_3)$ *the inequality*

$$u(t) \leq a(t)e^{\int_0^t f(s)ds} G^{-1}\Big\{G(\prod_{0<t_k<t}(1+\beta_k)) + \int_0^t g(s)\lambda(s,t)ds\Big\}, \tag{1.38}$$

holds, where function $\lambda(s,t)$ *is defined by equality (1.25), and*

$$\gamma_3 = \sup\big\{\xi \geq 0 : G(\prod_{0<t_k<t}(1+\beta_k)) + \int_0^t g(s)\lambda(s,t)ds \in Dom(G^{-1}) \quad \textit{for } t \in [0,\xi]\big\}.$$

Proof. We divide both parts of the inequality (1.37) by the function $a(t)$, use condition 3 and Theorem 1.2.1, and we obtain inequality (1.38). □

In the case when there is only a nonlinear part in the integrals in the right part of inequality (1.23) we obtain as a partial case of Theorem 1.2.1 the following result:

Corollary 1.2.3. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *Functions* $g(t) \in C([0,\infty),[0,\infty))$.
3. *Function* $u \in PC([0,\infty),[0,\infty))$ *and for* $t \geq 0$ *the inequality*

$$u(t) \leq a + \int_0^t g(s)Q(u(s))ds + \sum_{0<t_k<t} \beta_k u(t_k), \tag{1.39}$$

holds, where $a = const > 0$, $\beta_k = const \geq 0$, $k = 1, 2, \ldots$.

Then for $t \in [0, \gamma_4)$ *the inequality*

$$u(t) \leq G^{-1}\Big\{G(R(t)) + \int_0^t g(s)\big(\prod_{s<t_j<t} M_j\big)ds\Big\} \tag{1.40}$$

holds, where

$$\gamma_4 = \sup\big\{\xi \geq 0 : G(R(t)) + \int_0^t g(s)\big(\prod_{s<t_j<t} M_j\big)ds \in Dom(G^{-1}) \quad \textit{for } t \in [0,\xi]\big\}.$$

We will prove an impulsive integral inequality for piecewise continuous functions in the case when the unknown function and the integral are arguments of a nonlinear function.

Theorem 1.2.2. *Let the following conditions be satisfied:*

1. *Functions* $f(t), g(t)$, *and* $h(t) \in C([0,\infty),[0,\infty))$.
2. *Function* $F(t) \in C([0,\infty),[0,\infty))$ *is nondecreasing.*
3. *Function* $Q(t) \in C([0,\infty),[0,\infty))$ *is nondecreasing and there exists a function* $\varphi(t) \in C([0,\infty),[0,\infty))$ *such that* $Q(uv) \leq \varphi(u)Q(u)$ *for* $u, v \geq 0$.
4. *Function* $u(t) \in PC([0,\infty),[0,\infty))$ *satisfies for* $t \geq 0$ *the inequality*

$$u(t) \leq g(t)F\Big[c + \int_0^t h(s)Q(u(s))ds\Big] + f(t) \sum_{0<t_k<t} \beta_k u(t_k), \tag{1.41}$$

where $c = const \geq 0$, $\beta_k = const \geq 0$, $k = 1, 2, \ldots$.

Then for $t \in [0, \gamma_5)$ *the inequality*

$$\begin{aligned} u(t) \quad &\leq \quad \rho(t) \prod_{0<t_k<t} \big(1+\beta_k\rho(t_k)\big) F\Big\{H^{-1}\Big[H(c) \\ &+ \int_0^t h(s)\varphi\Big(\rho(s) \prod_{0<t_k<s} \big(1+\beta_k\rho(t_k)\big)\Big)ds\Big]\Big\} \end{aligned} \tag{1.42}$$

holds, where $\rho(t) = max\{g(t), f(t)\}$, H^{-1} *is the inverse function of the function* $H(u)$,

$$H(u) = \int_{u_0}^u \frac{ds}{Q(F(s))}, \quad u_0 > 0, \tag{1.43}$$

$$\begin{aligned} \gamma_5 \quad = \quad & \sup\Big\{\xi \geq 0 : H(c) + \int_0^t h(s)\varphi\Big(\rho(s) \prod_{0<t_k<s} \big(1+\beta_k\rho(t_k)\big)\Big)ds \\ & \in Dom(H^{-1}) \quad \text{for } t \in [0,\xi]\Big\}. \end{aligned}$$

Proof. We define function $v : [0,\infty) \to [0,\infty)$ by the equality

$$v(t) = c + \int_0^t h(s)Q(u(s))ds.$$

Function $v(t)$ is a nondecreasing differentiable function for $t \geq 0$, and satisfies the inequality

$$\begin{aligned} u(t) \quad &\leq \quad g(t)F(v(t)) + f(t) \sum_{0<t_k<t} \beta_k u(t_k) \\ &\leq \quad \rho(t)\Big[F(v(t)) + \sum_{0<t_k<t} \beta_k u(t_k)\Big]. \end{aligned} \tag{1.44}$$

Let $t \in (t_k, t_{k+1}] \bigcap [0, \gamma_5)$, $k \geq 0$, $t_0 = 0$. From the monotonicity of functions $v(t)$ and $F(u)$, and inequality (1.44) we obtain

$$\begin{aligned}
u(t) &\leq \rho(t)\Big[F(v(t)) + \sum_{i=1}^{k-1}\beta_i u(t_i) + \beta_k\rho(t_k)\Big(F(v(t_k)) + \sum_{i=1}^{k-1}\beta_i u(t_i)\Big)\Big] \\
&\leq \rho(t)\Big[(1+\beta_k\rho(t_k))\Big(F(v(t)) + \sum_{i=1}^{k-2}\beta_i u(t_i)\Big) \\
&\leq \rho(t)\Big\{(1+\beta_k\rho(t_k))\Big[F(v(t)) + \sum_{i=1}^{k-2}\beta_i u(t_i) \\
&\qquad +\beta_{k-1}\rho(t_{k-1})\Big(F(v(t_{k-1})) + \sum_{i=1}^{k-2}\beta_i u(t_i)\Big)\Big]\Big\} \\
&\leq \rho(t)(1+\beta_k\rho(t_k))(1+\beta_{k-1}\rho(t_{k-1}))\Big[F(v(t)) + \sum_{i=1}^{k-2}\beta_i u(t_i)\Big] \\
&\leq \cdots \leq \rho(t)\prod_{i=1}^{k}(1+\beta_i\rho(t_i))F(v(t)). \qquad (1.45)
\end{aligned}$$

We will note that $\sum_{i=1}^{k}\beta_i u(t_i) = 0$, $\prod_{i=1}^{k}(1+\beta_i\rho(t_i)) = 1$ for $k < 1$.

From the definition of function $v(t)$, inequality (1.45), and the properties of function $Q(u)$ follows that

$$\begin{aligned}
v'(t) &= h(t)Q(u(t)) \leq h(t)Q\Big\{\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))F(v(t))\Big\} \\
&\leq h(t)\varphi\Big(\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big)Q(F(v(t))). \qquad (1.46)
\end{aligned}$$

From inequality (1.46) and the definition of function $H(u)$ we obtain

$$\frac{d}{dt}\Big[H(v(t)\Big] = \frac{v'(t)}{Q(F(v(t)))} \leq h(t)\varphi\Big(\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big). \qquad (1.47)$$

We integrate inequality (1.47) from 0 to t, use the initial condition $v(0) = c$ and we obtain

$$H(v(t)) \leq H(c) + \int_0^t h(s)\varphi\Big(\rho(s)\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds. \qquad (1.48)$$

The inequalities (1.47) and (1.48) imply the validity of inequality (1.42). □

In the case when the nonlinear functions in the inequality (1.41) is multiplied by a constant, as a partial case of Theorem 1.2.2 we obtain the following result:

Corollary 1.2.4. *Let functions $h(t), Q(u), F(u)$ satisfy the conditions of Theorem 1.2.2 and function $u(t) \in PC([0,\infty),[0,\infty))$ satisfies for $t \geq 0$ the inequality*

$$u(t) \leq F\left[c + \int_0^t h(s)Q(u(s))ds\right] + \sum_{0<t_k<t}\beta_k u(t_k). \qquad (1.49)$$

Then for $t \in [0,\gamma_6)$ *the inequality*

$$u(t) \leq \prod_{0<t_k<t} (1+\beta_k) F\Big\{H^{-1}\Big[H(c)+\int_0^t h(s)\varphi\Big(\prod_{0,t_k<s}(1+\beta_k)ds\Big)\Big]\Big\}, \tag{1.50}$$

holds, where function $H(u)$ *is defined by equality* (1.43), H^{-1} *is the inverse function of* $H(u)$ *and*

$$\gamma_6 = \sup\Big\{\xi \geq 0 : H(c)+\int_0^t h(s)\varphi\Big(\prod_{0<t_k<s}(1+\beta_k)\Big)ds \in Dom(H^{-1}) \\ \text{for } t \in [0,\xi]\Big\}.$$

In the case when there is an additional function in the right part of the inequality (1.41), the main nonlinear function involved into the solution of the inequality is changed and the following result is proved:

Theorem 1.2.3. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, *and* 3 *of Theorem* 1.2.2 *are satisfied.*
2. *Function* $r(t) \in C([0,\infty),[0,\infty))$.
3. *Function* $u(t) \in PC([0,\infty),[0,\infty))$ *satisfies for* $t \geq 0$ *the inequality*

$$u(t) \leq r(t)+g(t)F\Big[c+\int_0^t h(s)Q(u(s))ds\Big]+f(t)\sum_{0<t_k<t}\beta_k u(t_k), \tag{1.51}$$

where $c = const \geq 0$, $\beta_k = const \geq 0$, $k = 1,2,\ldots$.

Then for $t \in [0,\gamma_7)$ *the inequality*

$$u(t) \leq \rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big\{1+F\Big[P^{-1}\big[P(c) \\ +\int_0^t h(s)\varphi\Big(\rho(s)\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds\big]\Big]\Big\} \tag{1.52}$$

holds, where $\rho(t) = max\{g(t), f(t), r(t)\}$, P^{-1} *is the inverse function of the function* $P(u)$,

$$P(u) = \int_{u_0}^{u} \frac{ds}{Q(1+F(s))}, \quad u_0 > 0, \tag{1.53}$$

$$\gamma_7 = \sup\Big\{\xi \geq 0 : P(c)+\int_0^t h(s)\varphi\Big(\rho(s)\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds \\ \in Dom(P^{-1}) \quad \text{for } t \in [0,\xi]\Big\}.$$

Proof. We define function $v:\ [0,\infty) \to [0,\infty)$ by the equality

$$v(t) = c+\int_0^t h(s)Q(u(s))ds. \tag{1.54}$$

Function $v(t)$ is nondecreasing differentiable and for $t \geq 0$ the inequality

$$u(t) \leq \rho(t)\Big\{1+F(v(t))+\sum_{0<t_k<t} \beta_k u(t_k)\Big\} \tag{1.55}$$

holds.

Let $t \in (t_k, t_{k+1}] \bigcap [0, \gamma_6)$, $k \geq 0$, $t_0 = 0$. From the monotonicity of the functions $v(t)$ and $F(u)$ and inequality (1.55) we obtain

$$\begin{aligned} u(t) \;&\leq\; \rho(t)\Big\{1+F(v(t))+\sum_{i=1}^{k-1}\beta_i u(t_i)+\beta_k\rho(t_k)\Big[1+F(v(t_k)) \\ &+\; \sum_{i=1}^{k-1}\beta_i u(t_i)\Big]\Big\} \leq \rho(t)\Big\{(1+\beta_k\rho(t_k))\Big(1+F(v(t))+\sum_{i=1}^{k-1}\beta_i u(t_i)\Big)\Big\} \\ &\leq\; \cdots \leq \rho(t)\Big(\prod_{i=1}^{k}(1+\beta_i\rho(t_i))\Big)\Big[1+F(v(t))\Big]. \end{aligned} \tag{1.56}$$

From the definition of function $v(t)$, inequality (1.56) and the properties of function $Q(u)$ follows that

$$\begin{aligned} v'(t) \;&=\; h(t)Q(u(t)) \leq h(t)Q\Big\{\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big(1+F(v(t))\Big)\Big\} \\ &\leq\; h(t)\varphi\Big(\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big)Q(1+F(v(t))). \end{aligned} \tag{1.57}$$

From inequality (1.57) and function $P(u)$ we obtain

$$\frac{d}{dt}\Big[P(v(t)\Big] = \frac{v'(t)}{Q(1+F(v(t)))} \leq h(t)\varphi\Big(\rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\Big). \tag{1.58}$$

We integrate inequality (1.58) from 0 to t, we use the initial condition $v(0) = c$ and obtain

$$P(v(t)) \leq P(c)+\int_0^t h(s)\varphi\Big(\rho(s)\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds. \tag{1.59}$$

Inequalities (1.56) and (1.59) imply the validity of inequality (1.52). □

Remark 2. We will note that all results in this section are true in the case when the lower bound 0 of the integrals is substituted by an arbitrary point $t_0 \geq 0$.

1.3. Impulsive Integral Inequalities for Piecewise Continuous Functions with a Delay of the Argument

In this section some integral inequalities for piecewise continuous functions with a constant delay of the argument are solved. The inequalities are generalizations of the classical integral inequalities of Gronwall-Bellman and Bihari. The importance of these impulsive inequalities is defined by their wide applications in qualitative investigations of impulsive differential-difference equations.

Initially we will solve a linear impulsive integral inequality for piecewise continuous functions with a constant delay of the argument.

Theorem 1.3.1. *Let the following conditions be fulfilled:*

1. *Functions* $f(t), r(t) \in C([0,\infty),[0,\infty))$.
2. *Function* $g(t) \in C([0,\infty),[0,\infty))$ *is nondecreasing.*
3. *Function* $\psi(t) \in C([-h,0],[0,\infty))$.
4. *Function* $u(t) \in PC([-h,\infty),[0,\infty))$ *satisfies the inequality*

$$u(t) \leq g(t) + \int_0^t f(s)u(s)ds + \int_0^t r(s)u(s-h)ds + \sum_{0<t_k<t} \beta_k u(t_k) \text{ for } t > 0, \tag{1.60}$$

$$u(t) \leq \psi(t), \quad t \in [-h,0], \tag{1.61}$$

where $h = const > 0,\ \ \beta_k = const \geq 0,\ \ k = 1,2,\ldots.$

Then for $t \geq 0$ *the inequality*

$$u(t) \leq \Big[g(t) + \int_0^{T(t)} r(s)\psi(s-h)ds\Big] \times \prod_{0<t_k<t} \big(1+\beta_k\big) e^{\int_0^t f(s)ds + \lambda(t)\int_h^t r(s)ds}, \tag{1.62}$$

holds, where

$$T(t) = \begin{cases} t & \text{for } t \in [0,h] \\ h & \text{for } t > h \end{cases}, \quad \lambda(t) = \begin{cases} 0 & \text{for } t \in [0,h] \\ 1 & \text{for } t > h \end{cases}. \tag{1.63}$$

Proof. Denote the right part of inequality (1.60) by $v(t)$. Function $v(t)$ is a nondecreasing piecewise continuous function, that has points of discontinuity $t_k,\ k = 1,2,\ldots$ and satisfies the inequalities

$$u(t) \leq v(t) \qquad \text{for } t \geq 0, \tag{1.64}$$

$$u(t-h) \leq v(t) \qquad \text{for } t \geq h. \tag{1.65}$$

Case 1. Let $t \in [0,h]$. From inequalities (1.60), (1.61), and (1.64) we obtain

$$v(t) \leq g(t) + \int_0^t f(s)\psi(s-h)ds + \int_0^t r(s)v(s)ds + \sum_{0<t_k<t} \beta_k v(t_k). \tag{1.66}$$

According to Lemma 1.1.2 for $t \in [0,h]$ the inequality

$$v(t) \leq \Big[g(t) + \int_0^t f(s)\psi(s-h)ds\Big]\Big(\prod_{0<t_k<t}(1+\beta_k)\Big)e^{\int_0^t r(s)ds} \tag{1.67}$$

holds.

Inequalities (1.64) and (1.67) prove the validity of inequality (1.62) for $t \in [0,h]$.

Case 2. Let $t \geq h$. Then there exists a natural number n such that $t_n \leq h < t_{n+1}$. From inequalities (1.60), (1.64), and (1.65) follows inequality

$$\begin{aligned} v(t) &= v(t_n) + [v(t) - v(t_n)] \\ &\leq v(t_n) + g(t) - g(t_n) + \int_{t_n}^{t} \big[f(s) + r(s)\big] v(s) ds + \sum_{t_n \leq t_k < t} \beta_k v(t_k) \\ &= (1 + \beta_n) v(t_n) + g(t) - g(t_n) + \int_{t_n}^{t} \big[f(s) + r(s)\big] v(s) ds \\ &\qquad + \sum_{t_n < t_k < t} \beta_k v(t_k). \end{aligned} \tag{1.68}$$

From inequality (1.68) according to Lemma 1.1.2 we obtain that inequality

$$v(t) \leq \Big[(1 + \beta_n) v(t_n) + g(t) - g(t_n)\Big] \Big(\prod_{t_n < t_k < t} (1 + \beta_k) \Big) e^{\int_{t_n}^{t} \big(f(s) + r(s)\big) ds} \tag{1.69}$$

holds for $t \geq h \geq t_n$.

From inequality (1.67) and the monotonicity of functions $v(t)$ and $g(t)$ we obtain

$$\begin{aligned} &(1 + \beta_n) v(t_n) + g(t) - g(t_n) = (1 + \beta_n) v(h) + g(t) - g(t_n) \leq \\ &\leq (1 + \beta_n) \Big\{ g(h) + \int_0^h f(s) \psi(s - h) ds \Big\} \Big(\prod_{k=1}^{n-1} (1 + \beta_k) \Big) e^{\int_0^h r(s) ds} \\ &\quad + [g(t) - g(h)] \Big(\prod_{k=1}^{n} (1 + \beta_k) \Big) e^{\int_0^h r(s) ds} \\ &= \Big[g(t) + \int_0^h f(s) \psi(s - h) ds \Big] \Big(\prod_{k=1}^{n} (1 + \beta_k) \Big) e^{\int_0^{t_n} r(s) ds}. \end{aligned} \tag{1.70}$$

From inequalities (1.64), (1.69), and (1.70) follows the validity of inequality (1.62) for $t \geq h$. □

We will prove some nonlinear impulsive integral inequalities for scalar piecewise continuous functions with a constant delay of the argument.

Definition 3. We will say that the function $G(u)$ belongs to the class W_1 if
1. $G \in C([0, \infty), [0, \infty))$.
2. $G(u)$ is a nondecreasing function.

Definition 4. We will say that the function $G(u)$ belongs to the class $W_2(\phi)$ if
1. $G \in W_1$.
2. For every $u, v \geq 0$ we have $G(uv) \leq \phi(u) G(v)$ where $\phi \in C([0, \infty), [0, \infty))$.

We note that if function $G \in W_1$ and it satisfies the inequality $G(uv) \leq G(u) G(v)$ for $u, v \geq 0$ then $G \in W_2$.

Theorem 1.3.2. *Let the following conditions be fulfilled:*
1. *Functions* $f_1, f_2, f_3, p, g \in C([0, \infty), [0, \infty))$.
2. *Function* $\psi \in C([-h, 0], [0, \infty))$.

3. *Function* $Q \in W_2(\varphi)$ *and* $Q(u) > 0$ *for* $u > 0$.
4. *Function* $G \in W_1$.
5. *Function* $u \in PC([-h,\infty),[0,\infty))$ *and it satisfies the inequalities*

$$\begin{aligned} u(t) &\leq f_1(t) + f_2(t)G\Big(c + \int_0^t p(s)Q(u(s))ds + \int_0^t g(t)Q(u(s-h))ds\Big) \\ &\quad + f_3(t)\sum_{0<t_k<t}\beta_k u(t_k) \quad \text{for } t \geq 0, \qquad (1.71) \\ u(t) &\leq \psi(t) \qquad\qquad \text{for } t \in [-h,0], \qquad (1.72) \end{aligned}$$

where $c \geq 0, \beta_k \geq 0, (k = 1,2,\dots)$.

Then for $t \in [0,\gamma)$ *we have the inequality*

$$\begin{aligned} u(t) &\leq \rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k)) \\ &\quad \times\Big\{1 + G\Big[H^{-1}\Big(H(A) + \int_0^t p(s)\varphi\Big(\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds \\ &\quad + \lambda(t)\int_h^t g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<s-h}(1+\beta_k\rho(t_k))\Big)ds\Big]\Big\}, \qquad (1.73) \end{aligned}$$

where

$$\lambda(t) = \begin{cases} 0 & \text{for } t \in [0,h], \\ 1 & \text{for } t > h, \end{cases}$$

$$\begin{aligned} &\rho(t) = max\{f_i(t) : i = 1,2,3\}, \quad A = c + hB_1Q(B_2), \\ &B_1 = max\{g(t) : t \in [0,h]\}, \quad B_2 = max\{\psi(t) : t \in [-h,0]\}, \end{aligned}$$

$$H(u) = \int_{u_0}^u \frac{ds}{Q(1+G(s))}, \quad u_0 \geq 0, \qquad (1.74)$$

$$\begin{aligned} \gamma &= \sup\{t \geq 0 : \int_0^\tau p(s)\varphi\Big(\prod_{0<t_k<s}(1+\beta_k\rho(t_k))\Big)ds \\ &\quad + \lambda(\tau)\int_h^\tau g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<s-h}(1+\beta_k\rho(t_k))\Big)ds \in Dom(H^{-1}) \\ &\quad \text{for } \tau \in [0,t]\}, \qquad (1.75) \end{aligned}$$

the function H^{-1} *is the inverse of* $H(u)$.

Proof. *Case* 1. Let $t_1 \geq h$.

Case 1.1. Let $t \in (0,h] \cap [0,\gamma) \neq \emptyset$.

From inequalities (1.71) and (1.72) follows that for $t \in (0,h] \cap [0,\gamma)$ the inequality

$$u(t) \leq \rho(t)\Big(1 + G\Big(A + \int_0^t p(s)Q(u(s))ds\Big)\Big) \qquad (1.76)$$

holds.

We define a function $v : [0,h]\cap[0,\gamma) \to [0.\infty)$ by the equality

$$v(t) = A + \int_0^t p(s)Q(u(s))ds. \tag{1.77}$$

The function $v_0^{(0)}(t)$ is a nondecreasing differentiable function over $[0,h]\cap[0,\gamma)$. Inequality (1.76) can be rewritten in the form

$$u(t) \le \rho(t)\Big(1+G(v_0^{(0)}(t))\Big). \tag{1.78}$$

Inequality (1.78), the definition (1.74) of the function $H(u)$, and the properties of function $Q(u)$ prove that

$$\begin{aligned} \frac{d}{dt}H(v_0^{(0)}(t)) &= \frac{(v_0^{(0)}(t))'}{Q\Big(1+G(v_0^{(0)}(t))\Big)} = \frac{p(t)Q(u(t))}{Q\Big(1+G(v_0^{(0)}(t))\Big)} \\ &\le \frac{p(t)Q\Big(\rho(t)\big(1+G(v_0^{(0)}(t))\big)\Big)}{Q\Big(1+G(v_0^{(0)}(t))\Big)} \le p(t)\varphi\big(\rho(t)\big). \end{aligned} \tag{1.79}$$

We integrate inequality (1.79) from 0 to t, where $t\in[0,h]\cap[0,\gamma)$, and we use $v_0^{(0)}(0) = A$ in order to obtain

$$H(v_0^{(0)}(t)) \le H(A) + \int_0^t p(s)\varphi\big(\rho(s)\big)ds. \tag{1.80}$$

Inequalities (1.78) and (1.80) imply the validity of inequality (1.73) for $t\in[0,h]\cap[0,\gamma)$.

Case 1.2. Let $t\in(h,t_1]\cap[0,\gamma)\neq\emptyset$.

From inequalities (1.71) and (1.72) we obtain

$$\begin{aligned} u(t) &\le \rho(t)\Bigg(1+G\Big(v_0^{(0)}(h)+\int_h^t p(s)Q(u(s))ds \\ &+ \int_h^t g(s)Q(u(s-h))ds\Big)\Bigg) \\ &= \rho(t)\Bigg(1+G\Big(v_0^{(1)}(t)\Big)\Bigg), \end{aligned} \tag{1.81}$$

where the function $v_0^{(1)} : [h,t_1]\cap[0,\gamma)\to[0,\infty)$ is defined by the equality

$$v_0^{(1)}(t) = v_0^{(0)}(h) + \int_h^t p(s)Q(u(s))ds + \int_h^t g(s)Q(u(s-h))ds.$$

From inequality (1.81), the definition of function $H(u)$, and the properties of function $Q(u)$ we obtain

$$\begin{aligned} \frac{d}{dt}H(v_0^{(1)}(t)) &= \frac{(v_0^{(1)}(t))'}{Q\Big(1+G(v_0^{(1)}(t))\Big)} = \frac{p(t)Q(u(t))+g(t)Q(u(t-h))}{Q\Big(1+G(v_0^{(1)}(t))\Big)} \\ &\le p(t)\varphi\big(\rho(t)\big)+g(t)\frac{Q\Big[\rho(t-h)\Big(1+G\Big(v_0^{(1)}(t-h)\Big)\Big)\Big]}{Q\Big(1+G(v_0^{(1)}(t))\Big)}. \end{aligned} \tag{1.82}$$

From the monotonicity of function $v_0^{(1)}(t)$, inequalities $v_0^{(0)}(t-h) \le v_0^{(0)}(h) \le v_0^{(1)}(t)$ for $h < t \le \min(2h, t_1)$, and inequality (1.82) follows the validity of the inequality

$$\frac{d}{dt} H(v_0^{(1)}(t)) \le p(t)\varphi\big(\rho(t)\big) + g(t)\varphi\big(\rho(t-h)\big).$$

We integrate the above inequality from 0 to t, where $t \in [0,h] \cap [0,\gamma)$, and we obtain

$$\begin{aligned} H(v_0^{(1)}(t)) &\le H(v_0^{(0)}(h)) + \int_h^t p(s)\varphi\Big(\rho(s)\Big)ds + \int_h^t g(s)\varphi\Big(\rho(s-h)\Big)ds \\ &\le H(A) + \int_0^t p(s)\varphi\Big(\rho(s)\Big)ds + \int_h^t g(s)\varphi\Big(\rho(s-h)\Big)ds. \end{aligned} \qquad (1.83)$$

Inequalities (1.81) and (1.83) prove the validity of inequality (1.73) for $t \in (h, t_1] \cap [0,\gamma)$.

Case 1.3. Let $t > t_1$. We will use mathematical induction to prove the inequality

$$u(t) \le \rho(t) \prod_{i=1}^{k} \Big(1 + \beta_i \rho(t_i)\Big)\Big(1 + G(v_k(t))\Big)$$
$$\text{for } \ t \in (t_k, t_{k+1}] \cap [0,\gamma),\ k \ge 1. \qquad (1.84)$$

Let $t \in (t_1, t_2] \cap [0,\gamma) \ne \emptyset$.

Define a function $v_1 : [t_1, t_2] \cap [0,\gamma) \to [0,\infty)$ by the equality

$$v_1(t) = v_0(t_1) + \int_{t_1}^t p(s)Q(u(s))ds + \int_{t_1}^t g(s)Q(u(s-h))ds,$$

where

$$v_0(t) = \begin{cases} v_0^{(0)}(t) & \text{for } t \in [0,h], \\ v_0^{(1)}(t) & \text{for } t \in (h, t_1]. \end{cases}$$

The function $v_1(t)$ is a nondecreasing differentiable function for $t \in (t_1, t_2] \cap [0,\gamma)$. It satisfies the inequality $v_1(t) \ge v_0(t_1)$ and

$$\begin{aligned} u(t) &\le \rho(t)\Big(1 + G\Big(v_1(t)\Big) + \beta_1 u(t_1)\Big) \\ &\le \rho(t)\Big(1 + G\Big(v_1(t)\Big) + \beta_1 \rho(t_1)\Big(1 + G(v_0(t_1))\Big)\Big) \\ &\le \rho(t)\Big(1 + G(v_1(t))\Big)\Big(1 + \beta_1 \rho(t_1)\Big). \end{aligned}$$

Therefore inequality (1.84) holds for $t \in (t_1, t_2] \cap [0,\gamma)$.

Assume that inequality (1.84) holds for $t \in (t_{k-1}, t_k] \cap [0,\gamma)$, where k is a natural number.

Let $t \in (t_k, t_{k+1}] \cap [0,\gamma)$.

Consider the function $v_k : [t_k, t_{k+1}] \cap [0,\gamma) \to [0,\infty)$, defined by the equality

$$v_k(t) = v_{k-1}(t_k) + \int_{t_k}^t p(s)Q(u(s))ds + \int_{t_k}^t g(s)Q(u(s-h))ds. \qquad (1.85)$$

The function $v_k(t)$ is nondecreasing differentiable function such that $v_k(t) \geq v_{k-1}(t_k)$. Then according to the inductive assumption we obtain for $t \in (t_k, t_{k+1}] \cap [0, \gamma)$ the following inequalities

$$\begin{aligned} u(t) &\leq \rho(t)\Big(1 + G\Big(v_k(t)\Big) + \beta_k u(t_k)\Big) \\ &\leq \rho(t)\Big(1 + G(v_k(t)) + \beta_k \rho(t_k)\Big(1 + G(v_{k-1}(t_k)) + \beta_{k-1} u(t_{k-1})\Big)\Big) \\ &\leq \rho(t)\Big(1 + G(v_k(t)) + \beta_{k-1} u(t_{k-1})\Big)\Big(1 + \beta_k \rho(t_k)\Big) \\ &\leq \cdots \leq \rho(t) \prod_{i=1}^{k} \Big(1 + \beta_i \rho(t_i)\Big)\Big(1 + G(v_k(t))\Big). \end{aligned}$$

Therefore inequality (1.84) holds for $t \in (t_k, t_{k+1}] \cap [0, \gamma), k \geq 2$.

We will use mathematical induction to prove the inequality

$$\begin{aligned} H(v_k(t)) &\leq H(A) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} p(s)\varphi\Big[\rho(s) \prod_{j=1}^{i-1} (1 + \beta_j \rho(t_j))\Big] ds \\ &\quad + \int_{t_k}^{t} p(s)\varphi\Big[\rho(s) \prod_{j=1}^{k} (1 + \beta_j \rho(t_j))\Big] ds \\ &\quad + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} g(s)\varphi\Big[\rho(s-h) \prod_{0<t_j<s-h} (1 + \beta_j \rho(t_k))\Big] ds \\ &\quad + \int_{t_k}^{t} g(s)\varphi\Big[\rho(s-h) \prod_{0<t_j<s-h} (1 + \beta_j \rho(t_k))\Big] ds \end{aligned} \tag{1.86}$$

for $t > t_1$.

Let $t \in (t_1, t_2] \cap [0, \gamma)$.

Consider the following two cases:

Case A1. Let $h \leq t_2 - t_1$ and $t \in (t_1 + h, t_2] \cap [0, \gamma)$. Then from inequality (1.84) we obtain

$$\begin{aligned} u(t-h) &\leq \rho(t-h)\Big(1 + G(v_1(t-h))\Big)\Big(1 + \beta_1 \rho(t_1)\Big) \\ &\leq \rho(t-h)\Big(1 + G(v_1(t))\Big)\Big(1 + \beta_1 \rho(t_1)\Big) \\ &= \rho(t-h)\Big(1 + G(v_1(t))\Big) \prod_{0<t_k<t-h} \Big(1 + \beta_k \rho(t_k)\Big). \end{aligned} \tag{1.87}$$

Case B1. Let $h > t_2 - t_1$ and $t \in (t_1, t_1 + h] \cap [0, \gamma)$. Then from (1.84) follows the validity of the inequality

$$u(t-h) \leq \rho(t-h)\Big(1 + G(v_0(t-h))\Big). \tag{1.88}$$

From inequalities (1.88) and $v_0(t-h) \le v_1(t)$ we obtain

$$\begin{aligned} u(t-h) &\le \rho(t-h)\Big(1+G(v_1(t))\Big) \\ &= \rho(t-h)\Big(1+G(v_1(t))\Big)\prod_{0<t_k<t-h}\Big(1+\beta_k\rho(t_k)\Big). \end{aligned} \tag{1.89}$$

Inequalities (1.84), (1.87), (1.89), the definition of function $v_1(t)$ and the properties of function $Q(u)$ imply that

$$\begin{aligned} v_1'(t) &= p(t)Q(u(t))+g(t)Q(u(t-h)) \\ &\le \Big(p(t)\varphi\Big(\rho(t)(1+\beta_1\rho(t_1))\Big) \\ &+ g(t)\varphi\Big(\rho(t-h)\prod_{0<t_k<t-h}\Big(1+\beta_k\rho(t_k)\Big)\Big) \\ &\quad \times Q\Big(1+G(v_1(t))\Big). \end{aligned}$$

From the above inequality and definition (1.74) of function $H(u)$ follows that

$$\begin{aligned} \frac{d}{dt}H(v_1(t)) &= \frac{(v_1(t))'}{Q\Big(1+G(v_1(t))\Big)} \\ &\le p(t)\varphi\Big(\rho(t)(1+\beta_1\rho(t_1))\Big) \\ &\quad +g(t)\varphi\Big(\rho(t-h)\prod_{0<t_k<t-h}\Big(1+\beta_k\rho(t_k)\Big)\Big). \end{aligned}$$

We integrate the above inequality from t_1 to t and use inequality (1.83) to obtain

$$\begin{aligned} H(v_1(t)) &\le H(v_0(t_1))+\int_{t_1}^{t} p(s)\varphi\Big(\rho(s)(1+\beta_1\rho(t_1))\Big)ds \\ &\quad +\int_{t_1}^{t} g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<s-h}\Big(1+\beta_k\rho(t_k)\Big)\Big)ds \\ &\le H(A)+\int_0^{t_1} p(s)\varphi\Big(\rho(s)\Big)ds+\int_0^{t_1} g(s)\varphi\Big(\rho(s-h)\Big)ds \\ &+ \int_{t_1}^{t} p(s)\varphi\Big(\rho(s)(1+\beta_1\rho(t_1))\Big)ds \\ &\quad +\int_{t_1}^{t} g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<s-h}\Big(1+\beta_k\rho(t_k)\Big)\Big)ds. \end{aligned}$$

Therefore inequality (1.86) holds for $t \in (t_1,t_2]\cap[0,\gamma)$.

Assume that inequality (1.86) holds for $t \in (t_{k-1},t_k]\cap[0,\gamma)$.

Let $t \in (t_k,t_{k+1}]\cap[0,\gamma)$.

Consider the following two cases:

Case Ak. Let $h \le t_{k+1} - t_k$ and $t \in (t_k + h, t_{k+1}] \cap [0, \gamma)$. Then from inequality (1.84) we obtain

$$\begin{aligned} u(t-h) &\le \rho(t-h)\Big(1 + G(v_k(t-h))\Big)\prod_{i=1}^{k}\Big(1+\beta_i\rho(t_i)\Big) \\ &\le \rho(t-h)\Big(1 + G(v_k(t))\Big)\prod_{i=1}^{k}\Big(1+\beta_i\rho(t_i)\Big) \\ &= \rho(t-h)\Big(1 + G(v_k(t))\Big)\prod_{0<t_i<t-h}\Big(1+\beta_i\rho(t_i)\Big). \end{aligned} \tag{1.90}$$

Case Bk. Let $h > t_{k+1} - t_k$ and $t \in (t_k, t_k + h] \cap [0, \gamma)$. Then

$$u(t-h) \le \rho(t-h)\Big(1 + G(v_{k-1}(t-h))\Big)\prod_{i=1}^{k-1}\Big(1+\beta_i\rho(t_i)\Big). \tag{1.91}$$

From inequalities (1.91) and $v_{k-1}(t-h) \le v_k(t)$ we obtain

$$\begin{aligned} u(t-h) &\le \rho(t-h)\Big(1 + G(v_k(t))\Big)\prod_{i=1}^{k-1}\Big(1+\beta_i\rho(t_i)\Big) \\ &= \rho(t-h)\Big(1 + G(v_k(t))\Big)\prod_{0<t_i<t-h}\Big(1+\beta_i\rho(t_i)\Big). \end{aligned} \tag{1.92}$$

Inequalities (1.84), (1.90), (1.92), equality (1.85), and the properties of function $Q(u)$ imply that

$$\begin{aligned} v_k'(t) &= p(t)Q(u(t)) + g(t)Q(u(t-h)) \\ &\le \Big(p(t)\varphi\Big(\rho(t)\prod_{i=1}^{k}\big(1+\beta_i\rho(t_i)\big)\Big) \\ &+ g(t)\varphi\Big(\rho(t-h)\prod_{0<t_i<t-h}\big(1+\beta_i\rho(t_i)\big)\Big)\Big) \\ &\quad \times Q\Big(1+G(v_k(t))\Big). \end{aligned} \tag{1.93}$$

From definition (1.74) and inequality (1.93) follows that

$$\begin{aligned} \frac{d}{dt}H(v_k(t)) &= \frac{(v_k(t))'}{Q\Big(1+G(v_k(t))\Big)} \\ &\le p(t)\varphi\Big(\rho(t)\prod_{i=1}^{k}(1+\beta_i\rho(t_i))\Big) \\ &\quad + g(t)\varphi\Big(\rho(t-h)\prod_{0<t_i<t-h}\Big(1+\beta_i\rho(t_i)\Big)\Big). \end{aligned}$$

We integrate the above inequality from t_k to t and use the inductive assumption to obtain

$$\begin{aligned} H(v_k(t)) &\le H(v_{k-1}(t_k)) + \int_{t_k}^{t} p(s)\varphi\Big(\rho(s)\prod_{i=1}^{k}(1+\beta_i\rho(t_i))\Big)ds \\ &\quad + \int_{t_k}^{t} g(s)\varphi\Big(\rho(s-h)\prod_{0<t_i<s-h}\Big(1+\beta_i\rho(t_i)\Big)\Big)ds \\ &\le H(A) + \int_0^{t_1} p(s)\varphi\Big(\rho(s)\Big)ds \\ &+ \int_{t_k}^{t} p(s)\varphi\Big(\rho(s)\prod_{i=1}^{k}(1+\beta_1\rho(t_1))\Big)ds \\ &\quad + \int_{t_k}^{t} g(s)\varphi\Big(\rho(s-h)\prod_{0<t_i<s-h}\Big(1+\beta_i\rho(t_i)\Big)\Big)ds. \end{aligned}$$

Therefore inequality (1.86) holds for $t > t_1$.

Inequalities (1.84) and (1.86) prove the validity of inequality (1.73) for $t > t_1$.

Case 2. There exists a natural number m such that $t_m \le h < t_{m+1}$. As in case 1, we use functions $v_k \in C\Big([t_k, t_{k+1}] \cap [0,\gamma), [0,\infty)\Big)$:

$$\begin{aligned} v_k(t) &= v_{k-1}(t_k) + \int_{t_k}^{t} p(s)Q(u(s))ds \ \text{ for } \ k = 0,1,\dots,m, \\ v_k(t) &= v_{k-1}(t_k) + \int_{t_k}^{t} p(s)Q(u(s))ds + \int_{t_k}^{t} g(s)Q(u(s-h))ds, k > m. \end{aligned} \tag{1.94}$$

to prove the validity of inequality (1.73). □

As a partial case of Theorem 1.3.2 we obtain the following result:

Corollary 1.3.5. *Let the conditions of Theorem* 1.3.2 *be fulfilled and the function* $\varphi(t)$ *satisfies the inequality* $\varphi(ts) \le \varphi(t)\varphi(s)$ *for* $t,s \ge 0$.

Then for $t \in (t_k, t_{k+1}] \cap [0,\gamma_3)$ *the inequality*

$$\begin{aligned} u(t) &\le \rho(t)\prod_{0<t_k<t}(1+\beta_k\rho(t_k)) \\ &\quad \times\Big\{1 + G\Big[H^{-1}\big(H(A) + \int_0^t p(s)\varphi\big(\rho(s)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))\big)ds \\ &\quad + \lambda(t)\int_h^t g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<t-h}(1+\beta_k\rho(t_k))\Big)ds\Big]\Big\} \end{aligned} \tag{1.95}$$

holds, where the functions $\lambda(t)$ *and* $H(u)$ *are the same as in Theorem 1.3.2 and*

$$\begin{aligned} \gamma_3 &= sup\{t \ge 0 : H(a) + \int_0^{\tau} p(s)\varphi(\rho(s)\prod_{0<t_k<t}(1+\beta_k\rho(t_k))ds \\ &\quad + \lambda(\tau)\int_h^{\tau} g(s)\varphi\Big(\rho(s-h)\prod_{0<t_k<t-h}(1+\beta_k\rho(t_k))\Big)ds \in Dom(H^{-1}) \\ &\quad \textit{for} \ \ \tau \in [0,t]\}. \end{aligned} \tag{1.96}$$

In the case when function $f_1(t) = 0$ in inequality (1.71), we obtain a different upper bound of the unknown piecewise continuous function, where the function $H(u)$ is defined in different way.

Theorem 1.3.3. *Let the following conditions be satisfied:*
1. *Functions* $f_2, f_3, p, g \in C([0,\infty),[0,\infty))$.
2. *Function* $\psi \in C([-h,0],[0,\infty))$.
3. *Function* $Q \in W_2(\varphi)$ *and* $Q(u) > 0$ *for* $u > 0$.
4. *Function* $G \in W_1$.
5. *Function* $u \in PC([-h,\infty),[0,\infty))$ *and it satisfies the inequalities*

$$\begin{aligned} u(t) &\leq f_2(t) G\Big(c + \int_0^t p(s)Q(u(s))ds + \int_0^t g(t)Q(u(s-h))ds\Big) \\ &\quad + f_3(t) \sum_{0<t_k<t} \beta_k u(t_k) \quad \text{for } t \geq 0, \qquad (1.97) \\ u(t) &\leq \psi(t) \quad \text{for } t \in [-h,0], \qquad (1.98) \end{aligned}$$

where $c \geq 0$, $\beta_k \geq 0, (k = 1,2,\ldots)$.

Then for $t \in (t_k, t_{k+1}] \cap [0,\gamma)$, $(k = 0,1,2,\ldots)$ *the inequality*

$$\begin{aligned} u(t) &\leq \rho(t)\prod_{i=1}^{k}(1+\beta_i\rho(t_i)) \\ &\times G\Big(H^{-1}\big\{H(A) + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(s)\varphi\big[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\big]ds \\ &+ \int_{t_k}^{t} p(s)\varphi\big[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\big]ds \\ &+ \lambda(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} g(s)\varphi\big[\rho(s-h)\prod_{0<t_j<s-h}(1+\beta_j\rho(t_k))\big]ds \\ &+ \lambda(t)\int_{t_k}^{t} g(s)\varphi\big[\rho(s-h)\prod_{0<t_j<s-h}(1+\beta_j\rho(t_k))\big]ds\big\}\Big) \qquad (1.99) \end{aligned}$$

holds, where the function $\lambda(t)$ *and the constants* A, B_1, B_2, γ *are defined in Theorem 1.3.2,* $\rho(t) = max\{f_i(t) : i = 2,3\}$,

$$H(u) = \int_{u_0}^{u} \frac{ds}{Q(G(s))}, \quad u_0 \geq A > 0. \qquad (1.100)$$

The proof of Theorem 1.3.3 is analogous to the proof of Theorem 1.3.2. □

As a partial case of Theorem 1.3.2 we obtain the following impulsive integral inequality for piecewise continuous functions without deviating argument.

Theorem 1.3.4. *Let the following conditions be fulfilled:*
1. *Conditions* 1, 3, 4 *of Theorem* 1.3.2 *are satisfied.*

2. *Function* $u \in PC([0,\infty),[0,\infty))$ *and satisfies the inequality*

$$u(t) \leq f_1(t) + f_2(t) G\Big\{c + \int_0^t p(s)Q(u(s))ds\Big\} + f_3(t) \sum_{0<t_k<t} \beta_k u(t_k) \quad \text{for } t \geq 0, \tag{1.101}$$

where $c \geq 0, \beta_k \geq 0, (k = 1,2,\dots)$.

Then for $t \in (t_k, t_{k+1}] \cap [0,\gamma_6),\ k = 0,1,2,\dots$ *the inequality*

$$\begin{aligned} u(t) \leq\ & \rho(t) \prod_{0<t_k<t} (1+\beta_k \rho(t_k)) \\ & \times \Bigg(1 + F\Big(P^{-1}\Big\{P(c) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} h(s)\varphi\big[\rho(s) \prod_{j=1}^{i-1}(1+\beta_j \rho(t_j))\big]ds \\ & + \int_{t_k}^{t} h(s)\varphi\big[\rho(s) \prod_{j=1}^{k}(1+\beta_j\rho(t_j))\big]ds\Big\}\Big)\Bigg) \end{aligned} \tag{1.102}$$

holds, where function $H(u)$ *is defined by* (1.74), *function* $\rho(t) = max\{f_i(t) : i = 1,2\}$,

$$\begin{aligned} \gamma_6 = \sup\Big\{ & t \geq 0 : H(c) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} p(s)\varphi\big[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\big]ds \\ & + \int_{t_k}^{\tau} p(s)\varphi\big[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\big]ds \in Dom(H^{-1}) \\ & \text{for } \tau \in (t_k,t_{k+1}] \cap [0,t],\ k = 0,1,\dots\Big\}. \end{aligned}$$

It is easy to see that inequality (1.102) in Theorem 1.3.4 gives us better estimate of the function $u(t)$ than inequality (1.52) in Theorem 1.2.3.

Remark 3. As a partial case of Theorems 1.3.2 , 1.3.3, 1.3.4 and Corollary 5, some integral inequalities for continuous functions, solved in [104], [105], [106], could be obtained.

We will study impulsive integral inequalities, in which the unknown function is in a power.

Theorem 1.3.5. *Let the following conditions are fulfilled:*

1. *Functions* $f,g,h,r \in C([0,\infty),[0,\infty))$.
2. *Function* $\psi \in C([-h,0],[0,\infty))$ *and* $\psi(t) \leq c$ *for* $t \in [-h,0]$, *where* $c \geq 0$.
3. *Function* $u \in PC([-h,\infty),[0,\infty))$ *and satisfies the inequality*

$$\begin{aligned} u^p(t) \leq\ & c + \int_0^t [f(s)u^p(s) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) \\ & + r(s)u(s-h)]ds + \sum_{0<t_k<t} \beta_k u^p(t_k) \quad \text{for } t \geq 0, \end{aligned} \tag{1.103}$$

$$u(t) \leq \psi(t) \quad \text{for } t \in [-h,0]. \tag{1.104}$$

where the constants $p > 1, 0 \leq q \leq p$.

Then for $t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots$ *the inequality*

$$u(t) \le \sqrt[p]{\prod_{i=1}^{k}(1+\beta_i)} \times \sqrt[p]{\Big(c + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds\Big) e^{\int_0^t \left(f(s)+g(s)+\frac{h(s)+r(s)}{p}\right)ds}} \tag{1.105}$$

holds.

Proof. *Case* 1. Let $t_1 \ge h$.

Case 1.1. Let $t \in (0,h]$. Define a function $v_0^{(0)} : [-h,h] \to [0,\infty)$ by the equalities

$$v_0^{(0)}(t) = \begin{cases} c + \int_0^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) \\ \qquad +r(s)u(s-h)]ds \quad \text{for } t \in [0,h], \\ \psi^p(t) \quad \text{for } t \in [-h,0). \end{cases}$$

The function $v_0^{(0)}(t)$ is nondecreasing on $[0,h]$. From the inequalities $u^p(t) \le v_0^{(0)}(t)$ and $x^m y^n \le mx + ny,\ \ m,n > 0,\ \ n+m = 1$ we obtain

$$u(t) \le \sqrt[p]{v_0^{(0)}(t)} \le \frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}, \quad t \in [0,h] \tag{1.106}$$

and

$$\begin{aligned} u(t-h) &\le \frac{\psi(t-h)}{p} + \frac{p-1}{p} \le \frac{c}{p} + \frac{p-1}{p} \\ &\le \frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}, \quad t \in [0,h]. \end{aligned} \tag{1.107}$$

From the definition of function $v_0^{(0)}(t)$ and inequalities (1.106) and (1.107) follows the validity of the inequality

$$\begin{aligned} (v_0^{(0)}(t))' &= f(t)u^p(t) + g(t)u^q(t)u^{p-q}(t-h) + h(t)u(t) \\ &\qquad +r(t)u(t-h) \\ &\le f(t)v_0^{(0)}(t) + g(t)\big(v_0^{(0)}(t)\big)^{\frac{q}{p}}\big(v_0^{(0)}(t-h)\big)^{\frac{p-q}{p}} \\ &\quad +h(t)\Big(\frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}\Big) + r(t)\Big(\frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}\Big) \\ &\le \Big(f(t) + g(t) + \frac{h(t)+r(t))}{p}\Big)v_0^{(0)}(t) \\ &\quad +(h(t)+r(t))\frac{p-1}{p} \end{aligned}$$

or

$$\begin{aligned} v_0^{(0)}(t) &\le v_0^{(0)}(0) + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds \\ &+ \int_0^t \Big(f(s)+g(s)+\frac{h(s)+r(s))}{p}\Big)v_0^{(0)}(s)ds. \end{aligned} \tag{1.108}$$

According to Lemma 1.1.1 and inequality (1.108) we obtain the inequality

$$v_0^{(0)}(t) \le \left(c+\frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right) \times e^{\left(\int_0^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right)}. \tag{1.109}$$

From inequality (1.109) follows the validity of inequality (1.105) for $t \in [0,h]$.
Case 1.2. Let $t \in (h,t_1]$.
Define a function $v_0^{(1)} : [h,t_1] \to [0,\infty)$ by the equality

$$v_0^{(1)}(t) = v_0^{(0)}(h)+\int_h^t [f(s)u^p(t)+g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s)+r(s)u(s-h)]ds.$$

From the definition of function $v_0^{(1)}(t)$ and inequality (1.103) we obtain

$$u^p(t) \le v_0^{(1)}(t), \ \ t \in (h,t_1]. \tag{1.110}$$

Consider the following two cases :
Case 1.2.1. Let $h < t \le min(t_1,2h)$. Then $t-h \in (0,h]$ and

$$u(t-h) \le \sqrt[p]{v_0^{(0)}(t-h)} \le \sqrt[p]{v_0^{(0)}(h)} \le \sqrt[p]{v_0^{(1)}(t)} \le \frac{v_0^{(1)}(t)}{p}+\frac{p-1}{p}. \tag{1.111}$$

Case 1.2.2. Let $t \in (2h,t_1]$. Then

$$u(t-h) \le \sqrt[p]{v_0^{(1)}(t-h)} \le \sqrt[p]{v_0^{(1)}(t)} \le \frac{v_0^{(1)}(t)}{p}+\frac{p-1}{p}. \tag{1.112}$$

From the definition of function $v_0^{(1)}(t)$ and inequalities (1.110), (1.111), and (1.112) we obtain the inequality

$$(v_0^{(1)}(t))' \le \left(f(t)+g(t)+\frac{h(t)+r(t))}{p}\right)v_0^{(1)}(t) +(h(t)+r(t))\frac{p-1}{p}. \tag{1.113}$$

From inequality (1.113) using Lemma 1.1.1 we obtain the following bound for function $v_0^{(1)}(t)$:

$$\begin{aligned}(v_0^{(1)}(t)) &\le \left(v_0^{(0)}(h)+\frac{p-1}{p}\int_h^t (h(s)+r(s))ds\right) \\ &\quad \times exp\left(\int_h^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right) \\ &\le \left(c+\frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right) \\ &\quad \times exp\left(\int_0^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right).\end{aligned} \tag{1.114}$$

Inequalities (1.110) and (1.114) prove the validity of inequality (1.105) on $(h,t_1]$.

Case 1.3. Let $t\in(t_1,t_2]$.

Define a function $v_1:[t_1,t_2]\to[0,\infty)$ by the equality

$$\begin{aligned} v_1(t) &= v_0(t_1)+\int_h^t [f(s)u^p(t)+g(s)u^q(s)u^{p-q}(s-h)+h(s)u(s) \\ &\quad +r(s)u(s-h)]ds+\beta_1 u^p(t_1), \end{aligned}$$

where

$$v_0(t)=\begin{cases} v_0^{(0)}(t) & \text{for } t\in[0,h], \\ v_0^{(1)}(t) & \text{for } t\in[h,t_1]. \end{cases}$$

We will note that the following inequalities $v_0(t)\le v_1(t)$, $u^p(t)\le v_1(t)$, $u^p(t_1)\le v_0(t_1)$, $u(t-h)\le\sqrt[p]{v_1(t)}$, $u(t-h)\le\sqrt[p]{v_1(t)}$ and $\sqrt[p]{v_1(t)}\le\frac{v_1(t)}{p}+\frac{p-1}{p}$ hold for $t\in(t_1,t_2]$.

Function $v_1(t)$ satisfies the inequality

$$\begin{aligned} v_1(t)\le &\le \Big((1+\beta_1)v_0(t_1)+\frac{p-1}{p}\int_{t_1}^t (h(s)+r(s))ds\Big) \\ &\quad \times exp\Big(\int_{t_1}^t \Big(f(t)+g(t)+\frac{h(s)+r(s)}{p}\Big)ds\Big) \\ &\le (1+\beta_1)\Big(c+\frac{p-1}{p}\int_0^t (h(s)+r(s))ds\Big) \\ &\quad \times exp\Big(\int_0^t \Big(f(t)+g(t)+\frac{h(s)+r(s)}{p}\Big)ds\Big). \end{aligned} \tag{1.115}$$

From the inequality $u^p(t)\le v_1(t)$ and (1.115) follows the validity of inequality (1.105) for $t\in(t_1,t_2]$.

By using mathematical induction we prove the validity of inequality (1.105) for $t\ge 0$.

Case 2. There exists a natural number m such that $t_m\le h<t_{m+1}$. As in the proof of case 1, we use functions $v_k(t), k=1,2,\ldots$ defined by the equalities

$$\begin{aligned} v_k(t) &= v_{k-1}(t_k)+\int_{t_k}^t [f(s)u^p(t)+g(s)u^q(s)u^{p-q}(s-h)+h(s)u(s) \\ &\quad +r(s)u(s-h)]ds+\beta_k u^p(t_k). \end{aligned}$$

□

Corollary 1.3.6. *Let condition* 1 *of Theorem* 1.3.5 *be fulfilled and*

$$u^p(t)\le c+\int_0^t [f(s)u^p(s)+h(s)u(s)]ds+\sum_{0<t_k<t}\beta_k u^p(t_k),\ \ \textit{for } t\ge 0,$$

where the constraints $p>1, 0\le q\le p$, *and* $u\in PC([0,\infty),[0,\infty))$.

Then, for $t\in(t_k,t_{k+1}]$, $k=0,1,2,\ldots$ *the inequality*

$$u(t)\le\sqrt[p]{\Big(\prod_{i=1}^k(1+\beta_i)\Big)\Big(c+\frac{p-1}{p}\int_0^t h(s)ds\Big)e^{\int_0^t\left(f(s)+\frac{h(s)}{p}\right)ds}}$$

holds.

1.4. Impulsive Integral Inequalities for Scalar Piecewise Continuous Functions of Two Variables

Impulsive integral inequalities for scalar functions of two variables can be applied for qualitative investigations of impulsive partial differential equations. For example, in the paper [95] the stability of the solutions of nonlinear impulsive partial differential equations is studied with the help of impulsive integral inequalities.

In this section some linear impulsive integral inequalities for scalar piecewise continuous functions of two variables are solved.

Let $\{x_i\}_0^\infty$ and $\{y_j\}_0^\infty$ be two increasing sequences of real numbers such that

$$\lim_{i\to\infty} x_i = \infty \quad \text{and} \quad \lim_{j\to\infty} y_j = \infty.$$

Let points $x_0 > 0, y_0 > 0$ be fixed. We denote by Ω the set of all scalar functions of two variables $u(x,y)$ such that:

(i) The function $u(x,y)$ is nonnegative and integrable on $G = \{(s,t) \in \mathbf{R}^2 : \; s \geq x_0, \; t \geq y_0\}$.

(ii) For any fixed number $y \geq y_0$ the function $u(x,y)$ is piecewise continuous in x, and it has points of discontinuity at points $\{x_i\}_0^\infty$, and $u(x_i,y) = \lim_{x\to x_i-0} u(x,y)$.

(iii) For any fixed number $x \geq x_0$ the function $u(x,y)$ is piecewise continuous function in y, and it has points of discontinuity at points $\{y_i\}_0^\infty$, and $u(x,y_i) = \lim_{y\to y_i-0} u(x,y)$.

We will prove some linear integral inequalities for piecewise continuous functions of two variables.

Theorem 1.4.1. *Let the following conditions be fulfilled:*

1. *The functions $u(x,y)$, $f(x,y)$ are from set Ω.*
2. *The inequality*

$$\begin{aligned} u(x,y) \quad &\leq \quad c + \int_{x_0}^{x} \int_{y_0}^{y} f(s,t)u(s,t)dsdt \\ &+ \sum_{\substack{x_0<x_i<x \\ y_0<y_j<y}} \beta_{ij} u(x_i,y_j) \quad \text{for } (x,y) \in G \end{aligned} \tag{1.116}$$

holds, where $\beta_{ij} = const \geq 0$.

Then for $(x,y) \in G$ the inequality

$$u(x,y) \leq c\Big(\prod_{\substack{x_0<x_i<x \\ y_0<y_j<y}} (1+\beta_{ij})\Big) e^{\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)dsdt} \tag{1.117}$$

holds.

Proof. We denote by $v(x,y)$ the right part of inequality (1.116). Function $v(x,y)$ is nondecreasing in x and y. For any fixed $y \geq y_0$ we define a function $w(x) = v(x,y)$. The function $w(x)$ is differentiable function for $x \geq x_0, \; x \neq x_i, i = 1,2,\ldots$ and satisfies the equalities

$$w'(x) = \int_{y_0}^{y} f(x,t)u(x,t)dt \quad \text{for } x \neq x_i, \tag{1.118}$$

$$w(x_i+0)-w(x_i)=\sum_{y_0<y_j<y}\beta_{ij}u(x_i,y_j)\quad i=1,2,\dots\ . \tag{1.119}$$

Since $u(x,t)\le v(x,t)\le v(x,y)=w(x)$ and $u(x_i,y_j)\le v(x_i,y_j)\le v(x_i,y)=w(x_i)$ for $y_0\le t\le y,\ \ y_0<y_j<y$, from the equalities (1.118) and (1.119) we obtain

$$w'(x)\le w(x)\int_{y_0}^{y}f(x,t)dt\quad \text{for } x\ne x_i,$$

$$w(x_i+0)-w(x_i)=\sum_{y_0<y_j<y}\beta_{ij}w(x_i)\quad i=1,2,\dots\ . \tag{1.120}$$

Inequality (1.120) implies the validity of the integral inequality

$$\begin{aligned} w(x)\ &\le\ w(x_0)+\int_{x_0}^{x}\Big\{\int_{y_0}^{y}f(s,t)dt\Big\}w(s)ds\\ &+\sum_{x_0<x_i<x}\Big\{\sum_{y_0<y_j<y}\beta_{ij}\Big\}w(x_i). \end{aligned}$$

From Lemma 1.1.2 and the above inequality follows the validity of the inequality

$$w(x)\le w(x_0)\prod_{x_0<x_i<x}\Big(1+\sum_{y_0<y_j<y}\beta_{ij}\Big)e^{\int_{x_0}^{x}\left(\int_{y_0}^{y}f(s,t)dt\right)ds}. \tag{1.121}$$

From inequality (1.121) and the relations

$$w(x_0)=c,\ \ 1+\sum_{j=1}^{m}\beta_{ij}\le\prod_{j=1}^{m}(1+\beta_{ij})$$

it follows the validity of the inequality

$$w(x)\le c\Big(\prod_{\substack{x_0<x_i<x\\ y_0<y_j<y}}(1+\beta_{ij})\Big)e^{\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)dsdt}. \tag{1.122}$$

From inequalities (1.122) and $u(x,y)\le v(x,y)=w(x)$ we obtain inequality (1.117). □

Theorem 1.4.2. *Let the following conditions be fulfilled:*

1. *Functions $u(x,y),f(x,y),a(x,y)$ are from set Ω, function $a(x,y)$ is decreasing in any of its arguments x and y.*
2. *For any point $(x,y)\in G$ the inequality*

$$u(x,y)\le a(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)u(s,t)dsdt+\sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}}\beta_{ij}u(x_i,y_j) \tag{1.123}$$

holds, where $\beta_{ij}=const\ge 0$, $x_0,y_0\in\mathbf{R}$ are fixed points.

Then for $(x,y) \in G$ *the inequality*

$$u(x,y) \leq a(x,y)\Big(\prod_{\substack{x_0<x_i<x\\ y_0<y_j<y}} (1+\beta_{ij})\Big)e^{\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)dsdt} \tag{1.124}$$

holds.

Proof. Let $(x,y) \in G$ and $y \geq y_0$ be fixed arbitrary points. Then for any couple (ξ,η) such that $x_0 \leq \xi \leq x$ and $y_0 \leq \eta \leq y$ we obtain from inequality (1.123) and the monotonicity of the function $a(x,y)$ the following inequality

$$u(\xi,\eta) \leq a(x,y) + \int_{x_0}^{\xi}\int_{y_0}^{\eta} f(s,t)u(s,t)dsdt + \sum_{\substack{x_0<x_i<\xi\\ y_0<y_j<\eta}} \beta_{ij}u(x_i,y_j). \tag{1.125}$$

From inequality (1.125) and Theorem 1.4.1 we obtain that for $x_0 \leq \xi \leq x$ and $y_0 \leq \eta \leq y$ the inequality

$$u(\xi,\eta) \leq a(x,y)\Big(\prod_{\substack{x_0<x_i<\xi\\ y_0<y_j<\eta}} (1+\beta_{ij})\Big)e^{\int_{x_0}^{\xi}\int_{y_0}^{\eta} f(s,t)dsdt} \tag{1.126}$$

holds.

Since $x_0 \leq \xi \leq x$ and $y_0 \leq \eta \leq y$, we can substitute $\xi = x$ and $\eta = y$ in inequality (1.126). Then we obtain inequality (1.124). □

Theorem 1.4.3. *Let the following conditions be fulfilled:*

1. *Functions* $u(x,y), f(x,y), a(x,y), g(x,y)$ *are from set* Ω.
2. *For* $(x,y) \in G$ *the inequality*

$$u(x,y) \leq a(x,y) + g(x,y)\Big\{\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u(s,t)dsdt + \sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}} \beta_{ij}u(x_i,y_j)\Big\} \tag{1.127}$$

holds, where $\beta_{ij} = const \geq 0$, $x_0, y_0 \in \mathbf{R}$ *are fixed points.*

Then for $(x,y) \in G$ *the inequality*

$$u(x,y) \leq a(x,y) + g(x,y)\Big\{\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)a(s,t)dsdt + \sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}} \beta_{ij}a(x_i,y_j)\Big\}$$

$$\times\Big(\prod_{\substack{x_0<x_i<x\\ y_0<y_j<y}}(1+\beta_{ij}g(x_i,y_j))\Big)e^{\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)g(s,t)dsdt} \tag{1.128}$$

holds.

Proof. We define the function

$$v(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)u(s,t)dsdt+\sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}}\beta_{ij}u(x_i,y_j). \tag{1.129}$$

From inequalities (1.127) and equality (1.129) we obtain

$$v(x,y)\le\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)a(s,t)dsdt+\sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}}\beta_{ij}a(x_i,y_j)$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)g(s,t)v(s,t)dsdt+\sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}}\beta_{ij}g(x_i,y_j)v(x_i,y_j). \tag{1.130}$$

According to Theorem 1.4.2 from inequality (1.130) follows the validity of the inequality

$$v(x,y)\quad\le\quad=\Big\{\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)a(s,t)dsdt+\sum_{\substack{x_0<x_i<x\\ y_0<y_j<y}}\beta_{ij}a(x_i,y_j)\Big\}$$

$$\times\Big(\prod_{\substack{x_0<x_i<x\\ y_0<y_j<y}}(1+\beta_{ij}g(x_i,y_j))\Big)e^{\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)g(s,t)dsdt}. \tag{1.131}$$

From inequalities (1.127), (1.131), and the definition of function $v(x,y)$ we obtain inequality (1.128). □

1.5. Applications of the Impulsive Integral Inequalities

With the help of some of the impulsive integral inequalities proved in this chapter we will study qualitative properties of the solutions of nonlinear impulsive equations. To show the wide specter of applications we will study various types of impulsive equations.

Case 1. **Impulsive ordinary differential equations**

Consider the nonlinear impulsive differential equation

$$x'=f(t,x),\quad t\neq t_k, \tag{1.132}$$

$$x(t_k+0)-x(t_k)=I_k(x(t_k)), \tag{1.133}$$

with initial condition

$$x(t_0)=x_0, \tag{1.134}$$

where $x\in\mathbf{R}^n$, $t_0\in\mathbf{R}$ is a fixed point.

The solution $x(t;t_0,x_0)$ of the initial value problem (1.132)–(1.134) satisfies the impulsive integral equation

$$x(t;t_0,x_0)=x_0+\int_{t_0}^{t} f(s,x(s;t_0,x_0))ds+\sum_{t_0<t_k<t} I_k(x(t_k;t_0,x_0)). \tag{1.135}$$

We will say that the conditions (H) are satisfied if:

H1. Function $f(t,x)\in C([t_0,\infty)\times\mathbf{R}^n,\mathbf{R}^n)$.

H2. Function $W(t,s)\in C([t_0,\infty)\times[0,\infty),[0,\infty))$ and satisfies the inequality $||f(t,x)||\leq W(t,||x||)$ for $t\geq t_0,\ x\in\mathbf{R}^n$.

H3. There exist functions $Q(t)\in C([0,\infty),[0,\infty))$, and $\lambda(t)\in C([t_0,\infty),[0,\infty))$ such that $\lambda(u)>0,\ \ u>0$ and $||f(t,x)-f(t,y)||\leq\lambda(t)Q(||x-y||)$ for $t\geq t_0,\ x,y\in\mathbf{R}^n$.

H4. For $(t_0,x_0)\in[0,\infty)\times\mathbf{R}^n$ the initial value problem (1.132)–(1.134) has a solution $x(t;t_0,x_0)$, defined for $t\geq t_0$.

H5. There exist functions $\delta_k\in C([0,\infty),[0,\infty)),\ k=1,2,\dots$ such that for $x\in\mathbf{R}^n$ the inequalities $||I_k(x)||\leq\delta_k(||x||),\ k=1,2,\dots$ hold.

H6. There exist functions $\gamma_k\in C([0,\infty),[0,\infty)), k=1,2,\dots$ such that for $x,y\in\mathbf{R}^n$ the inequalities $||I_k(x)-I_k(y)||\leq\gamma_k\big(||x-y||\big),\ \ k=1,2,\dots$ hold.

A/ Uniqueness of the solution of the initial value problem (1.132)–(1.134).

Let the conditions H1, H3, H4, H6 are fulfilled for $Q(x)=x$, $\gamma_k(x)=\beta_k x$, $\beta_k=const>0, k=1,2,\dots$.

Consider the function $u(t)=||x(t;t_0,x_0)-y(t;t_0,x_0)||$, where the functions $x(t;t_0,x_0)$ and $y(t;t_0,x_0)$ are two arbitrary solutions of the initial value problem (1.132)–(1.134).

Function $u(t)$ is nonnegative. From the integral equation (1.135), which is satisfied for both solutions $x(t;t_0,x_0)$ and $y(t;t_0,x_0)$, and the properties of the functions $f(t,x)$ and $I_k(x)$ follows the validity of the integral inequality

$$u(t)\leq\int_{t_0}^{t}\lambda(s)u(s)ds+\sum_{t_0<t_k<t}\beta_k u(t_k).$$

According to Theorem 1.1.1 for $c=0,\ p(t)=1,\ v(t)=\lambda(t)$ from the above inequality follows the inequality $u(t)\leq 0$, that proves the equality $u(t)=0$ for $t\geq t_0$, i.e. both solutions coincide.

B/ Continuous dependence of the solution of the initial value problem (1.132)–(1.134) *on the initial conditions.*

Let conditions H1, H3, H4 and H6 be fulfilled.

Consider the function $u(t)=||x(t;t_0,x_0)-y(t;t_0,y_0)||$, where the functions $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$ are two solutions of the initial value problem (1.132), (1.133), (1.134).

The function $u(t)$ is nonnegative.

From the integral equation (1.135), that is satisfied for both solutions $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$, and the properties of the functions $f(t,x)$ and $I_k(x)$ follows that the function $u(t)$ for $t \geq t_0$ satisfies the integral inequality

$$u(t) \leq ||x_0 - y_0|| + \int_{t_0}^{t} \lambda(s) Q(u(s)) ds + \sum_{t_0<t_k<t} \gamma_k\big(u(t_k)\big). \tag{1.136}$$

Let $Q(u) = u$ and $\gamma_k(u) = \beta_k u,\ \beta_k = const > 0,\ k = 1,2,\ldots$.

From inequality (1.136) according to Theorem 1.1.1 for $c = ||x_0 - y_0||$, $p(t) = 1$, $v(t) = \lambda(t)$ follows the inequality

$$u(t) \leq ||x_0 - y_0|| \prod_{t_0<t_i<t} (1+\beta_k) e^{\int_{t_0}^{t} \lambda(s) ds}, \quad t \geq t_0. \tag{1.137}$$

Let $\varepsilon > 0$ be an arbitrary number, $T > t_0$ be a fixed constant. We define a constant $\delta = \delta(\varepsilon) > 0$ by the equality

$$\delta = \varepsilon \Big[\prod_{t_0<t_i<T} (1+\beta_k) e^{K(T-t_0)} \Big]^{-1},$$

where $K = max\{\lambda(t) :\ t \in [t_0, T]\} < \infty$.

Then from inequality (1.137) follows that for $t \in [t_0, T]$ the inequality $u(t) < \varepsilon$ holds, i.e. the solutions of the initial value problem (1.132), (1.133), (1.134) depend continuously on the initial conditions. □

Remark 4. With the help of integral inequalities we can study the continuous dependence of the solutions on the parameter, on the impulsive functions, as well as the stability.

C/ Bounds of the solutions of the initial value problem (1.132)–(1.134).

We will obtain bounds of the solutions of the impulsive equation (1.132), (1.133) with different right parts.

Let conditions H1, H2, H4 and H5 be satisfied, where $W(t,u) = Lu,\ \gamma(u) = \beta_k u,\ L = const > 0,\ \beta_k = const \geq 0, k = 1,2,\ldots$.

From the integral equation (1.135), which the solution of the impulsive differential equation (1.132), (1.133) satisfies, the conditions of functions $f(t,x)$ and $I_k(x)$ and Theorem 1.1.1 we obtain the following bound for the solution $x(t;t_0,x_0)$ of the initial value problem (1.132)–(1.134):

$$||x(t;t_0,x_0)|| \leq ||x_0|| \prod_{t_0<t_k<t} (1+\beta_k) e^{L(t-t_0)}.$$

Let conditions H1, H2, H4 and H5 be fulfilled, where $W(t,u) = g(t)F(u)$, $\delta_k(u) = \beta_k u,\ \beta_k = const \geq 0,\ k = 1,2,\ldots$. Let function $g(t) \in C([t_0,\infty),[0,\infty))$, function $F(u)$ be nondecreasing, positive continuous function and there exist constants $M_k > 1,\ \ k = 1,2,\ldots,$ such that $G\Big((1+\beta_k)x\Big) - G\Big((1+\beta_k)y\Big) \leq M_k\Big(G(x) - G(y)\Big)$ for $x,y \in \mathbf{R}^n$ and $\int_{t_0}^{\infty} \Big(\prod_{s<t_i} M_i\Big) g(s) ds < \infty$, where $G(u) = \int_0^u \frac{ds}{F(s)}$.

Then according to Theorem 1.1.2 for $a = ||x_0||$, $f(s) \equiv 0$, we obtain the following bounds for the solution of the initial value problem for the impulsive differential equation (1.132)–(1.134):

$$||x(t;t_0,x_0)|| \leq G^{-1}\Big\{G(R(t)) + \int_{t_0}^{t}\Big(\prod_{s<t_i<t} M_i\Big)g(s)ds\Big\},$$

where $R(t) = ||x_0||\prod_{t_0<t_k<t}(1+\beta_k)$.

In the partial case, when the functions $g(t) \equiv L = const > 0$, $F(u) = \sqrt{u}$ and the constants $\beta_k = \beta \geq 0$, $(k = 1,2,\ldots)$, functions $G(u)$ and $G^{-1}(u)$ are defined by $G(u) = 2\sqrt{u}$ and $G^{-1}(u) = \frac{u^2}{4}$. In this case the constants $M_k = \sqrt{1+\beta}$, $k = 1,2,\ldots$ and we obtain the following bounds for the solution of the initial value problem for the impulsive differential equation (1.132)–(1.134):

$$||x(t;t_0,x_0)|| \leq \frac{(1+\beta)^k}{4}\Big\{2||x_0|| + L\Big\}^2 \text{ for } t \in (t_k,t_{k+1}],\ k = 0,1,2,\ldots.$$

Case 2. **Impulsive differential-difference equations**

We will illustrate some possible applications of impulsive integral inequalities for investigating of properties of the solutions of impulsive differential-difference equations.

Consider the nonlinear impulsive differential-difference equation

$$x' = f(t,x(t),x(t-h)), \quad t \geq t_0,\ t \neq t_k, \tag{1.138}$$

$$x(t_k+0) - x(t_k-0) = I(x(t_k)), \tag{1.139}$$

with initial condition

$$x(t) = \varphi(t), \quad t \in [t_0-h,t_0], \tag{1.140}$$

where $x \in \mathbf{R}^n, h = const > 0$, $t_0 \in \mathbf{R}$ is a fixed point.

The solution $x(t;t_0,\varphi)$ of the initial value problem (1.138)–(1.140) satisfies the equalities

$$\begin{aligned} x(t;t_0,\varphi) \quad &= \quad \varphi(t_0) + \int_{t_0}^{t} f(s,x(s;t_0,\varphi),x(s-h;t_0,\varphi))ds \\ &+ \sum_{t_0<t_k<t} I_k(x(t_k;t_0,\varphi)) \quad \text{for } t \geq t_0, \end{aligned} \tag{1.141}$$

$$x(t;t_0,\varphi) = \varphi(t) \ \text{ for } t \in [t_0-h,t_0]. \tag{1.142}$$

We will say that conditions (H) are satisfied if:

H1. Function $f(t,x,y) \in C([t_0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$.

H2. Function $\varphi(x) \in C([t_0-h,t_0],\mathbf{R}^n)$.

H3. There exists a function

$$W(t,s,r) \in C([t_0,\infty) \times [0,\infty) \times [0,\infty), [0,\infty))$$

such that $||f(t,x,y)|| \leq W(t,||x||,||y||)$ for $t \geq t_0$ and $x,y \in \mathbf{R}^n$.

H4. There exist two functions $Q(t,s) \in C([0,\infty) \times [0,\infty),[0,\infty))$ and $\lambda(t) \in C([t_0,\infty),(0,\infty))$ such that

$$||f(t,x_1,y_1) - f(t,x_2,y_2)|| \leq \lambda(t)Q(||x_1 - y_1||,||x_2 - y_2||)$$

for $t \geq t_0,\ x_i,y_i \in \mathbf{R}^n,\ i = 1,2.$

H5. For $t_0 \geq 0$, $\varphi \in C([t_0 - h,t_0],\mathbf{R}^n)$ the initial value problem (1.138)–(1.140) has a solution $x(t;t_0,\varphi)$, defined for $t \geq t_0 - h$.

H6. There exist functions $\delta_k(u) \in C([0,\infty),(0,\infty))$, $k = 1,2,\ldots$ such that for $x \in \mathbf{R}^n$ the inequalities $||I_k(x)|| \leq \delta_k(||x||), k = 1,2,\ldots$ hold.

H7. There exist functions $\gamma_k(u) \in C([0,\infty),(0,\infty))$, $k = 1,2,\ldots$ such that for $x,y \in \mathbf{R}^n$ the inequalities $||I_k(x) - I_k(y)|| \leq \gamma_k(||x - y||),\ \ k = 1,2,\ldots$ hold.

A/ Uniqueness of the solution of the initial value problem (1.138)–(1.140).

Let conditions H1, H2, H4, H5 and H7 be fulfilled for $Q(x,y) = x + y$, $\gamma_k(x) = \beta_k x, \beta_k = const > 0, k = 1,2\ldots.$

Consider the function $u(t) = ||x(t;t_0,\varphi) - y(t;t_0,\varphi)||$, where $x(t;t_0,\varphi)$ and $y(t;t_0,\varphi)$ are two arbitrary solutions of the initial value problem (1.138)–(1.140).

Function $u(t)$ is defined for $t \geq t_0 - h$ and it is nonnegative. From equalities (1.141), (1.142), that are satisfied for both solutions $x(t;t_0,\varphi)$ and $y(t;t_0,\varphi)$ and the properties of the functions $f(t,x,y)$ and $I_k(x)$ we obtain the integral inequality

$$\begin{aligned} u(t) \quad &\leq \quad \int_{t_0}^{t} \lambda(s)u(s)ds + \int_{t_0}^{t} \lambda(s)u(s-h)ds + \sum_{t_0 < t_k < t} \beta_k u(t_k) \quad \text{for } t \geq t_0, \\ & u(t) = 0 \quad \text{for } t \in [t_0 - h,t_0]. \end{aligned} \tag{1.143}$$

From the impulsive integral inequality (1.143) according to Theorem 1.2.1 we obtain that $u(t) \leq 0$ for $t \geq t_0$. Therefore both solutions $x(t;t_0,\varphi)$ and $y(t;t_0,\varphi)$ coincide.

B/ Continuous dependence of the solution of the initial value problem (1.138)–(1.140) *on the initial function.*

Let conditions H1, H2, H4, H5 and H7 be fulfilled.

We define a function $u(t) = ||x(t;t_0,\varphi) - y(t;t_0,\psi)||$, where $x(t;t_0,\varphi)$ and $y(t;t_0,\psi)$ are two solutions of the initial value problem (1.138)–(1.140).

Function $u(t)$ is nonnegative. From equalities (1.141), (1.142), that both solutions $x(t;t_0,\varphi)$ and $y(t;t_0,\psi)$ satisfy and the properties of the functions $f(t,x,y)$ and $I_k(x)$ follows the impulsive inequality for function $u(t)$

$$\begin{aligned} u(t) \quad &\leq \quad ||\varphi(t_0) - \psi(t_0)|| + \int_{t_0}^{t} \lambda(s)Q(u(s),u(s-h))ds \\ &\quad + \sum_{t_0 < t_k < t} \gamma_k u(t_k), \quad t \geq t_0, \end{aligned} \tag{1.144}$$

$$u(t) = ||\varphi(t) - \psi(t)||, \quad t \in [t_0 - h,t_0]. \tag{1.145}$$

Let $Q(u,v) = u+v$ and $\gamma_k(u) = \beta_k u,\ \beta_k = const > 0,\ k = 1,2,\ldots.$

From inequalities (1.144)and (1.145) according to Theorem 1.2.1 we obtain the inequality

$$u(t) \leq \Big\{||\varphi(t_0) - \psi(t_0)|| + \int_{t_0}^{T(t)} \lambda(s)||\varphi(s-h) - \psi(s-h)||ds\Big\}\times$$

$$\Big(\prod_{t_0<t_k<t}(1+\beta_k)\Big)e^{2\int_{t_0}^{t}\lambda(s)ds}, \quad t \geq t_0, \tag{1.146}$$

where $T(t) = \begin{cases} t, & \text{for } t \in [0,h] \\ h, & \text{for } t > h \end{cases}$.

Let $\varepsilon > 0$ be an arbitrary number, $T > t_0$ be a fixed constant. We define the number $\delta = \delta(\varepsilon) > 0$ by the equality

$$\delta = \varepsilon\Big[(1+Mh)\Big(\prod_{t_0<t_k<T}(1+\beta_k)\Big)e^{2M(T-t_0)}\Big]^{-1},$$

where $M = max\{\lambda(t) :\ t \in [t_0,h]\} < \infty.$

Let the initial functions $\varphi, \psi \in C([t_0-h,t_0],\mathbf{R}^n)$ be such that $sup\{||\varphi(t)-\psi(t)|| :\ t \in [t_0-h,t_0]\} < \delta.$ From the inequality (1.146) follows that for $t \in [t_0,T]$ the inequality $u(t) < \varepsilon$ holds, i.e. the solutions of the initial value problem (1.138)–(1.140) depend continuously on the initial function.

C/ Bounds for the solutions of the initial value problem (1.138)–(1.140).

We will consider impulsive differential-difference equations with different right parts.

Case C1. Let conditions H1, H2, H3, H5 and H6 be fulfilled, where $W(t,u,v) = Lu + Kv,\ \delta_k(u) = \beta_k u,\ L = const > 0, K = const > 0,\ \beta_k = const \geq 0, k = 1,2,\ldots.$

From equalities (1.141), (1.142), that are satisfied for the solutions of the initial value problem for the impulsive differential-difference equation (1.138)–(1.140), the properties of the functions $f(t,x,y)$ and $I_k(x)$, and Theorem 1.2.1 we obtain the following bound

$$||x(t;t_0,\varphi)|| \leq \Big[||\varphi(t_0)|| + K\int_{t_0}^{h}||\varphi(s)||ds\Big]\Big(\prod_{t_0<t_k<t}(1+\beta_k)\Big)e^{(L+K)(t-t_0)}.$$

Case C2. Let conditions H1, H2, H3, H5 and H6 be fulfilled, where

$$W(t,x,y) = g(t)F(x) + r(t)F(y),$$

$$\delta_k(u) = \beta_k u,\ \beta_k = const \geq 0, k = 1,2,\ldots.$$

Let functions $g(t), r(t) \in C([t_0,\infty),[0,\infty))$ are nondecreasing, function $F(u)$ is nondecreasing positive continuous and there exist constants $M_k > 1, k = 1,2,\ldots$ such that $G\Big((1+\beta_k)x\Big) - G\Big((1+\beta_k)y\Big) \leq M_k\Big(G(x) - G(y)\Big)$ for $x,y \in \mathbf{R}^n$ and the inequality $\int_0^\infty \Big(\prod_{s<t_i<t} M_i\Big)g(s)ds < \infty$ holds, where $G(u) = \int_0^u \frac{ds}{F(s)}$.

According to Theorem 1.2.3 we obtain the following bound for the solution of the initial value problem for the impulsive differential-difference equation (1.138)–(1.140):

$$\begin{aligned}||x(t;t_0,\varphi)|| &\leq G^{-1}\Big\{G\big(||\varphi(t_0)||\prod_{t_0<t_k<t}(1+\beta_k)\big)\\ &+ \int_{t_0}^{t}\Big(\prod_{s<t_i<t}M_i\Big)\Big(g(s)+r(s)\Big)ds\Big\},\end{aligned}$$

where $sup\{||\varphi(t):t\in[t_0-h,t_0]\}=||\varphi(t_0)||$.

Case C3. Let conditions H1, H2, H3, H5 and H6 be fulfilled, where function $W(t,u,v)=g(t)u^m+r(t)v^m$, $0<m<1$, $\delta_k(u)=\beta_k u$, $L=const>0$, $M=const>0$, $\beta_k=const\geq 0$, $k=1,2,\dots$.

Let functions $g(t),r(t)\in C([0,\infty),[0,\infty))$ be nondecreasing.

In this case the functions $G(u)$ and $G^{-1}(u)$ are defined by the equalities

$$G(u)=\int_0^u\frac{ds}{s^m}=\frac{s^{1-m}}{1-m}\quad\text{and}\quad G^{-1}(u)=\Big[(1-m)u\Big]^{\frac{1}{1-m}}.$$

Then $M_k=(1+\beta_k)^{1-m}$. The solution $x(t;t_0,\varphi)$ of the initial value problem for the impulsive differential-difference equation (1.138)–(1.140) satisfies the inequalities

$$\begin{aligned}||x(t;t_0,\varphi)|| &\leq ||\varphi(t_0)||+\int_{t_0}^{t}g(s)||x(s;t_0,\varphi)||^m ds\\ &+ \int_{t_0}^{t}r(s)||x(s-h;t_0,\varphi)||^m ds+\sum_{t_0<t_k<t}\beta_k||x(t_k;t_0,\varphi)||,\\ &\text{for } t\geq t_0,\end{aligned}\tag{1.147}$$

$$||x(t;t_0,\varphi)||=||\varphi(t)|| \text{ for } t\in[t_0-h,t_0].\tag{1.148}$$

Let the condition $sup\{||\varphi(t):t\in[t_0-h,t_0]\}=||\varphi(t_0)||$ be satisfied. Then according to Theorem 1.2.4 from inequalities (1.147), (1.148) follows the estimate for the solution of the initial value problem for the impulsive differential-difference equation (1.138)–(1.140)

$$\begin{aligned}||x(t;t_0,\varphi)|| &\leq \Big(\prod_{t_0<t_k<t}(1+\beta_k)\Big)\Big\{||\varphi(t_0)||^{1-m}\\ &+(1-m)\int_{t_0}^{t}\Big(g(s)+r(s)\Big)ds\Big\}^{\frac{1}{1-m}}.\end{aligned}$$

D/ Continuous dependence of the solutions of impulsive differential-difference equations on a parameter.

Consider the initial value problem for the nonlinear impulsive differential-difference equation that depends on a parameter:

$$x'=f(t,x(t),x(t-h),\lambda) \text{ for } t\geq t_0,\ t\neq t_k,\tag{1.149}$$

$$x(t_k+0)-x(t_k-0)=I(x(t_k),\lambda),\quad k=1,2,\dots,\tag{1.150}$$

$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0], \tag{1.151}$$

where $f : [t_0,\infty) \times \mathbf{R} \times \mathbf{R} \times \Lambda \to \mathbf{R}$, $\Lambda \subset \mathbf{R}$, $\varphi :\ [t_0 - h, t_0] \to \mathbf{R}$,$I_k :\ \mathbf{R} \times \Lambda \to \mathbf{R}, k = 1,2,\ldots,$ $h = const > 0$, t_0 is a fixed point, $\lambda \in \Lambda$ is a parameter.

We will prove the continuous dependence of the solution of (1.149)–(1.151) on the initial function and on the parameter.

Theorem 1.5.1. *Let the following conditions be fulfilled:*

1. *Function* $f \in C([t_0,\infty) \times \mathbf{R} \times \mathbf{R} \times \Lambda, \mathbf{R})$ *is Lipshitz, i.e. there exist positive constants* K, L, N *such that*

$$|f(t,x,y,\lambda) - f(t,\xi,\eta,\mu)| \le K|x-\xi| + L|y-\eta,| + N|\lambda-\mu|.$$

2. *Functions* $I_k \in C(\mathbf{R} \times \Lambda, \mathbf{R})$, $(k = 1,2,\ldots)$ *are Lipshitz, i.e. there exist positive constants* M_k, C_k *such that*

$$|I_k(x,\lambda) - I_k(\xi,\mu)| \le M_k|x-\xi| + C_k|\lambda-\mu|.$$

3. *The initial value problem for the nonlinear impulsive differential-difference equation* (1.149)–(1.151) *has a solution* $x(t)$ *for any* $x_0 \in \mathbf{R}, \lambda \in \Lambda$ *and* $\varphi \in PC([t_0 - h, t_0]$.

Then the solution $x(t)$ *of the initial value problem for the nonlinear impulsive differential-difference equation* (1.149)–(1.151) *depends continuously on the parameter, i.e. for any positive number* ε *there exists a positive number* $\delta = \delta(\varepsilon)$ *such that for every* $\lambda \in \Lambda$, *such that* $|\lambda - \lambda^*| < \delta$ *the inequality* $|x(t) - x^*(t)| < \varepsilon$ *holds for* $t \in [t_0, T]$, *where* $t_0 < T < \infty$, $x(t)$ *is the solution of* (1.149)–(1.151) *with a parameter* λ, *and* $x^*(t)$ *is the solution of* (1.149)–(1.151) *with a parameter* λ^*.

Proof. Let ε be an arbitrary positive number. We choose a number $\delta > 0$ such that

$$\delta < \frac{\varepsilon}{\Big[N(T-t_0) + \sum_{i:t_0<t_i<T} C_i\Big] \prod_{t_0<t_i<T}(1+M_i)} e^{-(K+L)(T-t_0)}.$$

Let $\lambda^* \in \Lambda$ and $\varphi \in PC([t_0 - h, t_0]$ be fixed. Let the parameter $\lambda \in \Lambda$ satisfies the inequality $|\lambda - \lambda^*| < \delta$. Then functions $x(t)$ and $x^*(t)$ satisfy the inequalities

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-h), \lambda)ds$$

$$+ \sum_{t_0<t_k<t} I_k(x(t_k),\lambda), \quad t \in [t_0, T], \tag{1.152}$$

$$x(t) = \varphi(t) \ \text{ for } t \in [t_0 - h, t_0], \tag{1.153}$$

and

$$x^*(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, x^*(s), x^*(s-h), \lambda^*)ds$$

$$+ \sum_{t_0<t_k<t} I_k(x^*(t_k),\lambda^*), \ t \in [t_0, T], \tag{1.154}$$

$$x^*(t) = \varphi(t) \ \text{ for } t \in [t_0 - h, t_0]. \tag{1.155}$$

The equalities (1.152)–(1.155) and the conditions 1 and 2 of Theorem 1.5.1 prove the validity of the impulsive integral inequality

$$\begin{aligned}
|x(t) - x^*(t)| \quad &\leq \quad \int_{t_0}^{t} K|x(s) - x^*(s)|ds + \int_{t_0}^{t} L|x(s-h) - x^*(s-h)|ds \\
&+ \quad N(T - t_0)\delta + \sum_{t_0 < t_k < t} M_k |x(t_k) - x*(t_k)| + \sum_{i: t_0 < t_i < T} C_i \delta \\
&\leq \quad \delta\Big[N(T - t_0) + \sum_{i: t_0 < t_i < T} C_i\Big] + \int_{t_0}^{t} K|x(s) - x^*(s)|ds \\
&+ \quad \int_{t_0}^{t} L|x(s-h) - x^*(s-h)|ds + \sum_{t_0 < t_k < t} M_k |x(t_k) - x^*(t_k)| \\
&\qquad \text{for } t \in [t_0, T].
\end{aligned}$$

The above inequality and Theorem 2.2.1 prove the continuous dependence of the solution of (1.149)–(1.151) on a parameter. □

Case 3. **Impulsive integral equations with delay**

Consider the scalar impulsive integral equation with delay

$$\begin{aligned}
u(t) \quad &= \quad f(t) + \Big[\int_0^t p(s)\sqrt{u(s)}ds + \int_0^t g(s)\sqrt{u(s-h)}ds\Big]^2 \\
&\qquad + \sum_{0 < t_k < t} \beta_k u(t_k) \quad \text{for} \quad t \geq 0,
\end{aligned} \tag{1.156}$$

$$u(t) = 0, \quad t \in [-h, 0], \tag{1.157}$$

where the functions $p, g \in C([0,\infty),[0,\infty))$, the function $f \in C([0,\infty),[0,1])$ $\beta_k \geq 0,\ k = 1,2,\ldots,$ and $h > 0$ are constants.

We will note that the solutions of (1.156), (1.157) are nonnegative.

We define the functions $G(u) = u^2, Q(u) = \sqrt{u}$. Then $Q \in W_2(\phi)$, where $\phi(u) = \sqrt{u}$ is a nondecreasing function.

Consider the function

$$H(u) = \int_0^u \frac{ds}{Q(1 + G(s))} = \int_0^u \frac{ds}{\sqrt{1 + s^2}} = ln(u + \sqrt{1 + u^2}). \tag{1.158}$$

Then the inverse function of the function $H(u)$ is defined by the equality

$$H^{-1}(u) = sh(u) = \frac{1}{2}(e^u - e^{-u}). \tag{1.159}$$

According to Corollary 5 we obtain the following bound for the function $u(t)$:

$$u(t) \leq \Big(\prod_{0 < t_k < t}(1 + \beta_k)\Big)\Big\{1 + \Big[sh\Big(\prod_{0 < t_k < t}(1 + \beta_k)\int_0^t (p(s) + g(s)ds)\Big]^2\Big\}.$$

Case 4. **Impulsive integro-differential equations**

Consider the initial value problem for the scalar nonlinear impulsive integro-differential equation

$$u'(t) = 2f(t)\sqrt{u(t)}\int_0^t f(s)\sqrt{u(s)}ds, \;\; t > 0, t \neq t_k, \tag{1.160}$$

$$u(t_k + 0) = \beta_k u(t_k), \tag{1.161}$$

$$u(0) = c, \tag{1.162}$$

where the function $f \in C([0,\infty),[0,\infty))$, and $c \geq 0, \beta_k \geq 0, k = 1,2,\ldots$ are constants.

The solutions of the above given problem satisfy the inequality

$$u(t) \leq c + \Big[\int_0^t f(s)\sqrt{u(s)}ds\Big]^2 + \sum_{0<t_k<t} \beta_k u(t_k), \;\; t_k \geq 0. \tag{1.163}$$

We define the functions $G(u) = u^2, Q(u) = \sqrt{u}$. Then the functions $H(u)$ and $H^{-1}(u)$ are defined by equalities (1.158) and (1.159).

We will note that the solutions of the initial value problem (1.160)–(1.162) are non-negative. According to Corollary 5 we obtain the following bound for the solutions of (1.160)–(1.162)

$$u(t) \leq A\Big(\prod_{0<t_k<t}(1+A\beta_k)\Big)\Big\{1 + \Big[sh\Big(\int_0^t \sqrt{A\prod_{0<t_k<s}(1+A\beta_k)F(s)ds}\Big)\Big]^2\Big\},$$

where $A = max(1,c)$.

Case 5. **Impulsive partial differential equations**

Let $\{x_i\}_0^\infty$ and $\{y_j\}_0^\infty$ be two increasing sequences of real numbers such that

$$\lim_{i\to\infty} x_i = \infty \qquad \text{and} \qquad \lim_{j\to\infty} y_j = \infty.$$

Let the scalar function $u(x,y)$ be from the set Ω, and the numbers a and b be fixed such that $x_0 \leq a, y_0 \leq b$.

Consider the nonlinear impulsive partial parabolic differential equation

$$u_{xy}(x,y) = F(x,y,u), \quad x \neq x_i, \;\; y \neq y_j, \tag{1.164}$$

$$u(x_i + 0, y_j + 0) = I_{ij}(u(x_i,y_j)), \quad i = 1,2,\ldots,p, \;\; j = 1,2,\ldots,m \tag{1.165}$$

with boundary conditions

$$u(x_0,y) = \varphi(y), \;\; u'_x(x,y_0) = \psi(x) \text{ for } x_0 \leq x \leq a, y_0 \leq y \leq b, \tag{1.166}$$

where $|F(x,y,u)| \leq f(x,y)|u|$, the function $\varphi(y)$ is nondecreasing, the functions $|I_{ij}(u)| \leq \beta_{ij}|u|$, β_{ij} are positive constants, $x_p < a \leq x_{p+1}, y_m < b \leq y_{m+1}$.

We will assume that the solution of (1.164), (1.165) , (1.166) exists on the rectangular $\{(x,u): \quad x_0 \leq x \leq a, y_0 \leq y \leq b\}$.

Then for $x_0 \leq x \leq a, y_0 \leq y \leq b$ the inequality

$$\begin{aligned} u(x,y) \quad &\leq \quad \varphi(y) + \int_{x_0}^{x} \psi(s)ds + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u(s,t)dsdt \\ &+ \sum_{\substack{x_0<x_i<x \\ y_0<y_j<y}} \beta_{ij} u(x_i, y_j) \end{aligned}$$

holds.

From the above inequality and Theorem 1.4.1 we obtain the following bound for the solution of the boundary value problem for the impulsive partial parabolic equation (1.164), (1.165) , (1.166)

$$u(x,y) \leq \Big(\varphi(y) + \int_{x_0}^{x} \psi(s)ds\Big)\Big(\prod_{\substack{x_0<x_i<x \\ y_0<y_j<y}} (1+\beta_{ij})\Big) e^{\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)dsdt}.$$

Chapter 2

Lyapunov's Method for Boundedness and Periodicity of the Solutions of Impulsive Equations

The following chapter discusses the boundedeness and the periodicity of the solutions of various types of impulsive equations. The main method of approach is a modification of the second method of Lyapunov and Razumikhin method. These methods are mainly applied for studying the stability of the solutions of various types of differential equations without impulses ([33], [37], [84], [85], [86], [87], [96], [122], [123]). Chapter 2 introduces particular types of piecewise continuous analogue of Lyapunov functions due to the existence of discontinuity of the solutions at the moments of impulses. Piecewise continuous functions and modification of the second method of Lyapunov are applied for studying the stability of the solutions of impulsive equations in [3], [18], [57], [58], [89], [100], [113], [114].

We note that similar results to the results obtained in this chapter are published in [67], [73], [74], [75], [77].

2.1. Piecewise Continuous Lyapunov Functions

Lyapunov functions are a powerful apparatus for qualitative investigations of differential equations. Since the solutions of the impulsive equations are discontinuous functions, the classical continuous Lyapunov functions are not applicable to impulsive equations. Therefore it is necessary to use piecewise continuous analogues of Lyapunov functions ([18], [89]).

Consider the general case of impulsive equations with variable moments of impulses, i.e. the impulses occur on the sets σ_k, $k = 1,2,3,\ldots$, defined by the equality (30), where the functions $\tau_k(x) \in C(\mathbf{R}^n,\mathbf{R})$, $\tau_k(x) < \tau_{k+1}(x)$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly on $x \in \mathbf{R}^n$.

Definition 5. We will say that the function $V(t,x) : [0,\infty) \times \mathbf{R}^n \to [0,\infty)$ is from the set W, if

1. Function $V(t,x)$ is continuous and differentiable on each of the sets G_k, $k = 1,2,\ldots$, $t \in [0,\infty)$, where the sets G_k are defined by (31).

2. For any integer k and $(\varsigma,\vartheta)\in\sigma_k$ the finite limits

$$V((\varsigma+0,\vartheta)) = \lim_{\substack{(t,x)\to((\varsigma,\vartheta))\\ (t,x)\in G_{k+1}}} V(t,x),$$

$$V((\varsigma,\vartheta)) = V((\varsigma-0,\vartheta)) = \lim_{\substack{(t,x)\to((\varsigma,\vartheta))\\ (t,x)\in G_{k}}} V(t,x)$$

exist, where the sets σ_k are defined by (30).

Remark 5. We will note that if point $(\varsigma,\vartheta)\notin\sigma_k$, then $V(\varsigma+0,\vartheta)=V(\varsigma,\vartheta)$.

With the help of piecewise continuous functions from the set W, we will obtain sufficient conditions for various types of boundedness of the solutions of impulsive equations. The defined piecewise continuous functions of the Lyapunov functions are used for impulsive differential equations, impulsive equations with "supremum", and impulsive hybrid equations in order to show their wide applicability. For any of these types of equations the derivative of the functions from set W along the trajectory of the solution is appropriately defined.

For the purpose of further investigations we consider the following sets of functions:

Definition 6. We will say that function $a(t)$ is from set CIP, if function $a\in C([0,\infty),[0,\infty))$, $a(0)=0$ and function $a(t)$ is increasing.

Definition 7. We will say that function $a(t)$ is from set Δ, if $a(t)$ is from set CIP and $\lim_{r\to\infty}a(r)=\infty$.

Definition 8. We will say that function $a(t)$ is from set Λ, if $a\in C([0,\infty),[0,\infty))$ and $a(s)>s,\ s\geq 0$.

2.2. Boundedness of the Solutions of Impulsive Differential Equations

Consider the initial value problem for impulsive differential equation with variable moments of impulses

$$x' = f(t,x),\quad t\neq\tau_k(x), \tag{2.1}$$

$$x(t+0)-x(t) = I_k(x(t)),\quad (t,x)\in\sigma_k,\ k=1,2\ldots, \tag{2.2}$$

$$x(t_0)=x_0, \tag{2.3}$$

where $x\in\mathbf{R}^n$, $(t_0,x_0)\in[0,\infty)\times\mathbf{R}^n$ is a fixed point.

The solution of the initial value problem of the system of impulsive differential equations (2.1), (2.2), (2.3) is denoted by $x(t;t_0,x_0)$.

Let function $V(t,x)$ be from set W. Define derivative along the trajectory of a solution of the impulsive differential equation (2.1), (2.2) by the following equality

$$D^-V_{(2.1),(2.2)}(t,x) = \limsup_{\varepsilon\to 0-}(1/\varepsilon)\{V(t+\varepsilon,x(t+\varepsilon;t,x))-V(t,x)\} \tag{2.4}$$

for $(t,x) \notin \sigma_k,\ k = 1,2,\ldots.$

We will say that conditions **H** are satisfied if:

H1. Functions $\tau_k : \mathbf{R}^n \to [0,\infty),\ \ (k = 1,2,\ldots)$ are continuous, $\tau_{k+1}(x) > \tau_k(x)$ for $x \in \mathbf{R}^n$ and the inequality

$$inf\{\tau_k(x) - \tau_{k-1}(x);\ k \geq 2, x \in \mathbf{R}^n\} > 0$$

holds.

H2. There exists $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly on $x \in \mathbf{R}^n$.

H3. The integral curve of the solution of the system of impulsive differential equations (2.1), (2.2) meets any of the sets σ_k at most once.

We will note that some sufficient conditions for absence of the phenomenon "beating" are given in [41].

Consider the following sets

$$B_\alpha = \{x \in \mathbf{R}^n :\ ||x|| \leq \alpha\},$$

$$\Omega_H = \{x :\ ||x|| \geq H\},$$

$$S_\alpha = \left\{(t,x) :\ \begin{array}{c} (t,x) \in [0,\infty) \times B_\alpha \ \text{ for } (t,x) \in \bigcup_{i=1}^{\infty} G_i, \\ (t, x + I_k(x)) \in [0,\infty) \times B_\alpha \ \text{ for } (t,x) \in \sigma_k \end{array}\right\}.$$

We will give the definitions of the basic types of boundedness of the solutions of the impulsive differential equations, that are analogues of those given in [123] for ordinary differential equations.

Definition 9. The solution $x(t;t_0,x_0)$ of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) is called *bounded*, if there exists a constant $\beta(t_0,x_0) > 0$ such that for $t \geq t_0$ the inequality $||x(t;t_0,x_0)|| < \beta(t_0,x_0)$ holds.

Definition 10. The solutions of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) are called *equi bounded*, if for any constant $\alpha > 0$ and for all $t_0 \geq 0$, a constant $\beta = \beta(t_0,\alpha) > 0$ exists such that for $x_0 \in B_\alpha$ and $t \geq t_0$ the inequality $||x(t;t_0,x_0)|| < \beta$ holds.

Definition 11. The solutions of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) are called *uniformly bounded*, if for any constant $\alpha > 0$ and for all $t_0 \geq 0$, there exists a constant $\beta = \beta(\alpha) > 0$ such that for $x_0 \in B_\alpha$ and $t \geq t_0$ the inequality $||x(t;t_0,x_0)|| < \beta$ holds.

Definition 12. The solutions of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) are called *ultimately bounded*, if there exists a constant $B > 0$ such that for $(t_0,x_0) \in S_\alpha$ there exists a constant $T = T(t_0,x_0) > 0$ such that for $t \geq t_0 + T$ the inequality $||x(t;t_0,x_0)|| < B$ holds.

Definition 13. The solutions of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) are called *equi-ultimately bounded* with a bound B, if for any constant $\alpha > 0$ and for all $t_0 \geq 0$, there exists a constant $T = T(t_0,\alpha) > 0$ such that for $x_0 \in B_\alpha$ and $t \geq t_0 + T$ the inequality $||x(t;t_0,x_0)|| < B$ holds.

Definition 14. The solutions of the initial value problem for impulsive differential equations (2.1), (2.2), (2.3) are called *uniform-ultimately bounded* with a bound B, if for any constant $\alpha > 0$ there exists a constant $T = T(\alpha) > 0$ such that for all $x_0 \in B_\alpha$ and $t \geq t_0 + T$ the inequality $||x(t;t_0,x_0)|| < B$ holds.

We will obtain the relationship between different types of boundedness of the solutions of impulsive differential equations.

Lemma 2.2.1. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *Functions* $I_k \in C(\mathbf{R}^n, \mathbf{R}^n)$, $k = 1,2\dots.$
3. *Function* $f \in C([0,\infty) \times \mathbf{R}^n, \mathbf{R}^n)$ *is a Lipshitz in its second argument.*
4. *The solution* $x(t;t_0,x_0)$ *of the initial value problem for the system of impulsive differential equations* (2.1), (2.2), (2.3) *is defined for* $t \in [t_0,T]$, *where* $(t_0,x_0) \in [0,\infty) \times \mathbf{R}^n$ *is a fixed point,* $T = const > t_0$.

Then for any number $\alpha > 0$ *a positive constant* $\beta = \beta(t_0,\alpha)$ *exists such that if* $(t_0,x_0) \in S_\alpha$, *then for* $t \in [t_0,T]$ *the inequality* $||x(t;t_0,x_0)|| < \beta$ *holds.*

Proof. From conditions H2 and H3 the integral curve $(t,x(t;t_0,x_0))$ for $t \in [t_0,T]$ intersects the sets σ_k, $k = 1,2,\dots$ finite number times. Let the points at which the integral curve intersects sets σ_k are $\xi_1 < \xi_2 < \dots < \xi_p$, where $\xi_k = \tau_{j_k}(x(\xi_k;t_0,x_0))$, $k = 1,2,\dots,p$, $p < \infty$. From the continuity of the solution $x(t;t_0,x_0)$ on $[t_0,\xi_1]$ it follows that there exists a constant $\beta(t_0,\alpha) > 0$ such that $||x(t;t_0,x_0)|| < \beta$ for $t \in [t_0,\xi_1]$. From condition 2 of Lemma 2.2.1 follows that there exists a constant $K_1 = K_1(\beta) > 0$ such that for $||x|| < \beta$ the inequality $||I_{j_1}(x)|| < K_1$ holds. Then $||x(\xi_1+0)|| = ||x(\xi_1) + I_{j_1}(x(\xi_1))|| \leq \beta + K_1$, where $x(t) = x(t;t_0,x_0)$. Therefore there exists a constant $\nu = \nu(t_0,\alpha) > 0$ such that $||x(t;t_0,x_0)|| = ||x(t;\xi_1+0,x(\xi_1+0;t_0,x_0))|| \leq \nu$ for $t \in (\xi_1,\xi_2]$. Similar to the above given proof we can show that on each interval $(\xi_i,\xi_{i+1}]$, $i = 1,2,\dots$ Lemma 2.2.1 is true. □

Lemma 2.2.2. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2 *and* 3 *of Lemma* 2.2.1 *are satisfied.*
2. *The solutions of the initial value problem for the system of impulsive differential equations* (2.1), (2.2), (2.3) *are equi-ultimately bounded with a bound B.*

Then the following conclusions are true:

(i) *the solutions of the initial value problem for the system of impulsive differential equations* (2.1), (2.2), (2.3) *are equi bounded;*

(ii) *for any two numbers* $\alpha > 0$ *and* $\xi > 0$ *there exists a number* $T = T(\xi,\alpha) > 0$ *such that if* $(t_0,x_0) \in S_\alpha$ *and* $t_0 \in [0,\xi]$, *then the inequality* $||x(t;t_0,x_0)|| < B$ *holds for* $t \geq t_0 + T$.

Proof. (i). Let $\alpha > 0$ be a fixed number. Choose a point $(t_0,x_0) \in S_\alpha$. According to condition 2 of Lemma 2.2.2 there exists a number $T = T(t_0,\alpha) > 0$ such that for $t \geq t_0 + T$ the inequality $||x(t;t_0,x_0)|| < B$ holds. From Lemma 2.2.1 it follows that there exists a number $\beta = \beta(t_0,\alpha) > 0$ such that $||x(t;t_0,x_0)|| < \beta$ for $t \in [t_0,t_0+T]$. Introduce the notation $\gamma(t_0,\alpha) = max(\beta,B)$. Then for $t \geq t_0$ the inequality $||x(t;t_0,x_0)|| < \gamma$ holds.

(ii). Let $\alpha > 0$ and $\xi > 0$ be fixed numbers and $t_0 \in [0,\xi]$. From condition 2 of Lemma 2.2.2 it follows that there exists a number $T_1 = T_1(\xi,\alpha) > 0$ such that $||x(t;t_0,x_0)|| =$

$||x(t;\xi,x(\xi;t_0,x_0))|| < B$ for $t \geq \xi + T_1$, where $(t_0,x_0) \in S_\alpha$. Introduce the notation $T = \xi + T_1$, $T = T(\xi,\alpha) > 0$. Then for $t \geq t_0 + T$, $(t_0,x_0) \in S_\alpha$ the inequality $||x(t;t_0,x_0)|| < B$ holds. □

We will study the boundedness of the solutions of the impulsive differential equations with the help of functions from the set W.

In further considerations we will use the following result:

Lemma 2.2.3. *Let conditions* H1 *and* H2 *be fulfilled, and function* $V(t,x)$ *be from set* W.

Then for any positive number α *and for all points* $t_0 \in [0,\infty)$ *there exists a number* $K = K(t_0,\alpha) > 0$ *such that* $V(t_0,x) \leq K$ *for* $||x|| < \alpha$.

Proof. Assume that the conclusion is not true, i.e. there exist a number $\alpha > 0$ and points $x_i \in \mathbf{R}^n$ such that $x_i \neq x_k$ for $i \neq k$, $x_i \in B_\alpha$ and

$$V(t_0,x) \geq i, \quad i = 1,2,\ldots \tag{2.5}$$

The sequence $\{x_i\}_1^\infty$ is bounded and therefore there exists a bounded subsequence $\{x_{i_k}\}_1^\infty$, such that $\lim_{k\to\infty} x_{i_k} = \beta$.

Case 1. Let there exists a natural number k such that $(t_0,\beta) \in G_k$, i.e. $\tau_{k-1}(\beta) < t_0 < \tau_k(\beta)$. From the continuity of functions $\tau_k(x)$ it follows that $\lim_{i\to\infty} \tau_{k-1}(x_i) = \tau_{k-1}(\beta)$ and $\lim_{i\to\infty} \tau_k(x_i) = \tau_k(\beta)$. Since $(t_0,\beta) \in G_k$ we obtain that for enough large integer i the inclusion $(t_0,x_i) \in G_k$ holds. Then the equality $\lim_{i\to\infty} V(t_0,x_i) = V(t_0,\beta)$ holds, that contradicts the inequality (2.5). The obtained contradiction proves Lemma 2.2.3.

Case 2. Let there exists a natural number k such that $(t_0,\beta) \in \sigma_k$, i.e. $\tau_k(\beta) = t_0$. From the continuity of functions $\tau_k(x)$ it follows that $\lim_{i\to\infty} \tau_k(x_i) = \tau_k(\beta) = t_0$, $\lim_{i\to\infty} \tau_{k+1}(x_i) = \tau_{k+1}(\beta) > \tau_k(\beta) = t_0$, and $\lim_{i\to\infty} \tau_{k-1}(x_i) = \tau_{k-1}(\beta) < t_0$. Therefore there exist infinite number of elements x_{i_j}, $j = 1,2,\ldots$ of the sequence $\{x_i\}_1^\infty$, such that one of the following inclusions is true:

(i) $x_{i_j} \in G_k$, $j = 1,2,\ldots$

or

(ii) $x_{i_j} \in G_{k+1}$, $j = 1,2,\ldots$

Then we obtain

$$\lim_{j\to\infty} V(t_0,x_{i_j}) = \begin{cases} V(t_0 - 0,\beta) & \text{for } x_{i_j} \in G_k,\ j = 1,2,\ldots, \\ V(t_0 + 0,\beta) & \text{for } x_{i_j} \in G_{k+1},\ j = 1,2,\ldots \end{cases}$$

The above equality and inequality (2.5) contradict the condition that the limits $V(t_0 + 0,\beta)$ and $V(t_0 - 0,\beta)$ are finite. The obtained contradiction proves Lemma 2.2.3. □

Theorem 2.2.1. *Let the following conditions be fulfilled:*

1. *Conditions (H) are satisfied.*
2. *Functions* $I_k \in C(\mathbf{R}^n,\mathbf{R}^n)$, $k = 1,2\ldots.$
3. *Function* $f \in C([0,\infty) \times \mathbf{R}^n,\mathbf{R}^n)$ *is Lipshitz in its second argument.*
4. *The solution of the initial value problem of the system of impulsive differential equations* (2.1), (2.2), (2.3) *is defined for* $t \geq t_0$, *where* $(t_0,x_0) \in [0,\infty) \times \mathbf{R}^n$ *is an arbitrary point.*
5. *There exists a function* $V(t,x)$ *from set* W *with the following properties*

(i) $a(||x||) \le V(t,x)$ *for* $(t,x) \in [0,\infty) \times \mathbf{R}^n$, *where* $a \in \Delta$;
(ii) $D^{-}_{(2.1),(2.2)}V(t,x) \le 0$ *for* $(t,x) \in \bigcup_{k=1}^{\infty} G_k$;
(iii) $V(t+0, x+I_k(x)) \le V(t,x)$ *for* $(t,x) \in \sigma_k, \ \ k = 1,2,\ldots$.

Then the solutions of the initial value problem for the system of impulsive differential equations (2.1), (2.2), (2.3) *are equi bounded.*

Proof. Let $\alpha > 0$ be an arbitrary number and $(t_0, x_0) \in S_\alpha$.

Case 1. Let $(t_0,x_0) \in \bigcup_{k=1}^{\infty} G_k$.

According to Lemma 2.2.3 there exists a constant $K = K(t_0,\alpha) > 0$ such that $V(t_0,x) \le K$ for $||x|| < \alpha$. We choose a number $\beta > 0$ such that $a(\beta) > K$.

Assume that there exists a number $\rho \ge t_0$ such that $||x(\rho;t_0,x_0)|| \ge \beta$. Then inequalities

$$a(\beta) \le a(||x(\rho;t_0,x_0)||) \le V(\rho, ||x(\rho;t_0,x_0)||) \le V(t_0,x_0) \le K$$

hold.

The obtained contradiction proves Theorem 2.2.1.

Case 2. Let $(t_0,x_0) \in \sigma_k$ for a natural number k. Then $||x_0 + I_k(x_0)|| \le \alpha$ and $V(t_0 + 0, x_0 + I_k(x_0)) \le K$. As in the proof of the case 1 we obtain a contradiction.

Therefore $||x(t;t_0,x_0)|| < \beta$ for $t \ge t_0$. □

Theorem 2.2.2. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*

2. *Functions* $I_k \in C(\mathbf{R}^n, \mathbf{R}^n), \ \ k = 1,2\ldots$ *and for* $x \in B_H$ *the inclusion* $x + I_k(x) \in B_H$ *holds, where* $H = const > 0$.

3. *Function* $f \in C([0,\infty) \times \mathbf{R}^n, \mathbf{R}^n)$ *is Lipshitz in its second argument.*

4. *The solution of the initial value problem of the system of impulsive differential equations* (2.1), (2.2), (2.3) *is defined for* $t \ge t_0$, *where* $(t_0,x_0) \in [0,\infty) \times \mathbf{R}^n$ *is an arbitrary point.*

5. *There exists a function* $V(t,x)$ *from set* W *with properties*

(i) $a(||x||) \le V(t,x) \le b(||x||)$ *for* $(t,x) \in [0,\infty) \times \Omega_H$, *where* $a \in \Delta$, $b \in CIP$;
(ii) $D^{-}_{(2.1),(2.2)}V(t,x) \le 0$ *for* $(t,x) \in \bigcup_{k=1}^{\infty} G_k$;
(iii) $V(t+0, x+I_k(x)) \le V(t,x)$ *for* $(t,x) \in \sigma_k, \ \ k = 1,2,\ldots$.

Then the solutions of the initial value problem for the system of impulsive differential equations (2.1), (2.2), (2.3) *are uniformly bounded.*

Proof. Let $(t_0,x_0) \in S_\alpha$, where $\alpha > 0$ is an arbitrary number. We will prove that for $t \ge t_0$ the inequality $||x(t;t_0,x_0)|| < H$ holds.

Let point x_0 be such that $H \le ||x_0|| < \alpha$. We choose a number $\beta = \beta(\alpha) > 0$ such that $b(\alpha) < a(\beta)$. Assume that there exists a number $\varsigma \ge t_0$ such that $||x(\varsigma;t_0.x_0)|| \ge \beta$. From condition 2 of Theorem 2.2.2 follows that there exist points $\eta, \xi \ge t_0$ such that $H \le ||x(\eta;t_0.x_0)|| \le \alpha$, $||x(\xi;t_0,x_0)|| \ge \beta$, $(\xi, x(\xi;t_0,x_0) \in \bigcup_{k=1}^{\infty} G_k$ and $||x(t;t_0,x_0)|| \ge H$ for $t \in [\eta,\xi]$. From condition 5 of Theorem 2.2.2 we obtain the inequality

$$a(\beta) \le V(\xi, x(\xi;t_0,x_0)) \le V(\eta, x(\eta;t_0,x_0)) \le b(\alpha),$$

that contradicts the choice of the point β. □

Theorem 2.2.3. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*

2. *Functions* $I_k \in C(\mathbf{R}^n, \mathbf{R}^n)$, $k = 1, 2\dots$ *and for* $x \in B_H$ *the inclusion* $x + I_k(x) \in B_H$ *holds, where* $H = const > 0$.

3. *Function* $f \in C([0,\infty) \times \mathbf{R}^n, \mathbf{R}^n)$ *is Lipshitz in its second argument.*

4. *The solution of the initial value problem for the system of impulsive differential equations* (2.1), (2.2), (2.3) *is defined for* $t \geq t_0$, *where* $(t_0, x_0) \in [0,\infty) \times \mathbf{R}^n$ *is an arbitrary point.*

5. *There exists a function* $V(t,x)$ *from set* W *with properties*

(i) $a(||x||) \leq V(t,x) \leq b(||x||)$ *for* $(t,x) \in [0,\infty) \times \Omega_H$, *where* $a \in \Delta$, $b \in CIP$ *;*

(ii) $D^-_{(2.1),(2.2)} V(t,x) \leq -c(||x||)$ *for* $(t,x) \in \bigcup_{k=1}^{\infty} G_k \cap \Omega_H$, *where the function* $c(x) \in C([0,\infty),(0,\infty))$*;*

(iii) $V(t+0, x+I_k(x)) \leq V(t,x)$ *for* $(t,x) \in \sigma_k \cap \Omega_H$, $k = 1, 2, \dots$.

Then the solutions of the initial value problem for system of impulsive differential equations (2.1), (2.2), (2.3) *are uniform-ultimately bounded.*

Proof. Let $(t_0, x_0) \in S_\alpha$, where $\alpha > 0$ is an arbitrary number. According to Theorem 2.2.2 the solutions of the initial value problem for system of impulsive differential equations (2.1), (2.2), (2.3) are uniformly bounded, i.e. there exists a number $\beta = \beta(\alpha) > 0$ such that $||x(t;t_0,x_0)|| < \beta$ for $t \geq t_0$. Therefore the inequality $\beta > \alpha \geq H$ holds. From the uniform boundedness of the solutions of (2.1), (2.2), (2.3) follows that there exists a constant $B > 0$ such that for $(t_0, y) \in S_H$ and $t \geq t_0$ the inequality $||x(t;t_0,y)|| < B$ holds. Assume that $||x(t;t_0,x_0)|| > H$ for $t \geq t_0$. From conditions (ii) and (iii) follows the existance of a number $\gamma = \gamma(\alpha) > 0$ such that for $H \leq ||x|| \leq \beta$ and $t \geq t_0$ the inequality $V(t,x) \leq V(t_0,x_0) - \gamma(t-t_0)$ holds. The last inequality, condition (i) and the assumption $||x(t;t_0,x_0)|| \geq H$ for $t \geq t_0$ imply the validity of the inequalities

$$a(H) \leq V(t, x(t;t_0,x_0)) \leq V(t_0,x_0) - \gamma(t-t_0) \leq b(\alpha) - \gamma(t-t_0). \tag{2.6}$$

We denote $T = T(\alpha) = \frac{b(\alpha)-a(H)}{\gamma} > 0$. Consider inequality (2.6) for $t > t_0 + T$. The obtained contradiction proves that the assumption is false. Therefore there exists a number $\varsigma \leq t_0 + T$ such that $||x(\varsigma;t_0,x_0)|| < H$.

If $(t_0+T, x(t_0+T;t_0,x_0)) \in \bigcup_{k=1}^{\infty} G_k$, then there exists a number $\nu \in [t_0, t_0+T)$ such that for $t \geq \nu$ the inequality $||x(t;t_0,x_0)|| < B$ holds.

If $(t_0+T, x(t_0+T;t_0,x_0)) \in \sigma_k$ for a natural number k, then there exists small enough number $\varepsilon > 0$ such that $||x(t_0+T_1;t_0,x_0)|| \leq H$ where $T_1 = T(\alpha) + \varepsilon$. Then for $t \geq t_0 + T_1$ the inequality $||x(t;t_0,x_0)|| < B$ holds. □

We will obtain sufficient and necessary conditions for equi-ultimate boundedness of the solutions of the initial value problem for system of impulsive differential equations (2.1), (2.2), (2.3) in the case when the moments of impulses are fixed, i.e. we consider the initial value problem for the impulsive differential equations

$$x' = f(t,x), \quad t \neq t_k, \tag{2.7}$$

$$x(t_k+0) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2\dots, \tag{2.8}$$

$$x(t_0) = x_0, \tag{2.9}$$

where $x \in \mathbf{R}^n$, the points $t_k: \ t_k < t_{k+1}, \ k = 0,1,\ldots, \ \lim_{k\to\infty} t_k = \infty, \ inf\{t_{k+1} - t_k: \ k = 1,2,\ldots\} > 0$ are fixed.

Theorem 2.2.4. *Let the following conditions be fulfilled:*

1. *Functions* $I_k \in C(\mathbf{R}^n, \mathbf{R}^n), \quad k = 1,2\ldots$ *are Lipshitz and for* $x \in B_H$ *the inclusion* $x + I_k \in B_H$ *is satisfied, where* $H = const > 0$.

2. *Function* $f \in C([0,\infty)\times\mathbf{R}^n, \mathbf{R}^n)$ *is Lipshitz in its second argument.*

3. *The solution of the initial value problem for the system of impulsive differential equations* (2.7), (2.8), (2.9) *is defined for* $t \ge t_0$, *where* $(t_0, x_0) \in [0,\infty)\times\mathbf{R}^n$ *is an arbitrary point.*

Then the necessary and sufficient condition for equi ultimate boundedness of the solutions of the initial value problem for the system of impulsive differential equations (2.7), (2.8), (2.9) *is the existence of a function* $V(t,x)$ *from the set* W *with the properties*

(i) $a(||x||) \le V(t,x)$ *for* $t \ge 0$, $x \in \Omega_H$, *where* $a \in \Delta$;

(ii) $D^-_{(2.7),(2.8)}V(t,x) \le -cV(t,x)$ *for* $t \ne t_k$, $k = 1,2,\ldots, x \in \Omega_H$, *where* $c = const > 0$;

(iii) $V(t_k + 0, x + I_k(x)) \le V(t_k, x)$ *for* $x \in \Omega_H, \ \ k = 1,2,\ldots$.

Proof. *Sufficiency.* Let $(t_0, x_0) \in S_\alpha$, where $\alpha > 0$ is an arbitrary number. According to Lemma 2.2.3 there exists a number $K = K(t_0,\alpha) > 0$ such that $V(t_0,x_0) \le K$. We choose a number

$$T = T(t_0,\alpha) \le \frac{1}{c}\ln\Big(\frac{K}{a(B)}\Big).$$

From properties (ii) and (iii) follows that for $t \ge t_0$ the inequalities

$$V(t, x(t;t_0,x_0)) \le V(t_0,x_0)e^{-c(t-t_0)} < a(B) \tag{2.10}$$

hold.

From property (i) and inequalities (2.10) follows the validity of inequality

$$a(||x(t;t_0,x_0)||) \le V(t,x(t;t_0,x_0)) < a(B). \tag{2.11}$$

Inequality (2.11) implies that $||x(t;t_0,x_0)|| < B$ for $t \ge t_0 + T$.

Necessity. Let the solutions of the initial value problem for system of impulsive differential equations (2.7), (2.8), (2.9) be equi ultimately bounded with a bound B'.

We define the function $a(u): \mathbf{R} \to [0,\infty)$ by the equality

$$a(u) = \begin{cases} u - B' & \text{for } u \ge B' \\ 0 & \text{for } 0 \le u < B' \end{cases}.$$

Let $(t,x) \in [0,\infty)\times\mathbf{R}^n$ be an arbitrary point. We will assume that $t \in (t_i, t_{i+1}]$.

We define functions

$$V_j(t,x) = \sup_{\eta \ge 0}\{a(||x(t+\eta;t,x)||)e^{c\eta}: \ t+\eta \in (t_{i+j}, t_{i+j+1}]\}, \quad j = 0,1,\ldots$$

and

$$V(t,x) = \sup_{j\ge 0} V_j(t,x).$$

We will prove that function $V(t,x)$ is from set W.

Let $(t,x),(t',x') \in [0,\infty)\times \mathbf{R}^n$ be arbitrary points such that $t,t' \in (t_k,t_{k+1}]$, $t<t'$. There exists a constant $\alpha>0$ such that $x,x' \in B_\alpha$.

From the equi boundedness of the solutions follows that there exist two constants $T = T(t,\alpha)>0$ and $T'=T'(t',\alpha)>0$ such that

$$||x(\xi;t,x)|| < B' \text{ for } \xi > t+T \quad \text{and} \quad ||x(\xi;t',x')|| < B' \text{ for } \xi > t'+T.$$

Therefore $a(||x(\xi;t,x)||)=0$ and $a(||x(\xi;t',x')||)=0$ for $\xi > t'+max(T,T')$. Then the following properties are satisfied:

(*j*) there exists a natural number p such that $V(t,x)=\max_{k\le j\le k+p} V_j(t,x)$ and $V(t',x') = \max_{k\le j\le k+p} V_j(t',x')$;

(*jj*) there exist a natural number $j:\ k\le j\le k+p$ such that $V(t,x)=V_j(t,x)$.

From the definition of the function $V(t,x)$ follows that inequality $V(t',x') \ge V_j(t',x')$ holds.

Case 1. Let $V_j(t,x)=a(||x(t_{j+1};t,x)||)e^{c(t_{j+1}-t)}$. The inequalities

$$V_j(t',x') \ge a(||x(t_{j+1};t',x')||)e^{c(t_{j+1}-t')} \ge a(||x(t_{j+1};t',x')||)e^{c(t_{j+1}-t)}$$

hold.

Therefore the following inequalities are true:

$$\begin{aligned} V(t,x)-V(t',x') &\le \Big\{a(||x(t_{j+1};t,x)||)-a(||x(t_{j+1};t',x')||)\Big\}e^{c(t_{j+1}-t)} \\ &\le \Big\{||x(t_{j+1};t,x)||-||x(t_{j+1};t',x')||\Big\}e^{c(t_{j+1}-t)} \\ &\le e^{c(t_{j+1}-t)}\Big\{\Big[||x(t_{j+1};t,x)||-||x(t';t,x)||\Big] \\ &\quad +\Big[||x(t';t,x)||-||x(t;t,x)||\Big] \\ &\quad +\Big[||x(t;t,x)||-||x(t';t',x')||\Big] \\ &\quad +\Big[||x(t';t',x')||-||x(t_{j+1};t',x')||\Big]\Big\}. \end{aligned}$$

From conditions 1 and 2 of Theorem 2.2.4 follows the existence of constants $M,L>0$ such that

$$V(t,x)-V(t',x') \le e^{c\gamma}\Big\{L|t-t'|+N||x-x'||\Big\}, \tag{2.12}$$

where $\gamma = inf\{t_{k+1}-t_k:\ k=1,2,\ldots\}$.

Case 2. There exists a point $\xi \in (t_j,t_{j+1})$ such that

$$V_k(t,x)=a(||x(\xi;t,x)||)e^{c(\xi-t)}.$$

As in the case 1 we can prove the validity of the inequality (2.12).

Therefore for $(t,x)\in[0,\infty)\times\mathbf{R}^n, t\ne t_k, k=1,2\ldots$ function $V(t,x)$ is continuous in both of its arguments and it is Lipshitz in its second argument.

We will prove that property (i) is satisfied for function $V(t,x)$.

Indeed, let $(t,x) \in [0,\infty) \times \Omega_{B'}$ be an arbitrary point. We assume that $t \in (t_i, t_{i+1}]$. Function $V(t,x)$ satisfies the inequalities

$$V(t,x) \geq V_i(t,x) \geq a(||x(t;t,x)||) = a(||x||).$$

Function $a(r) \geq 0$ for $r \geq B'$ is continuous and

$$\lim_{r\to\infty} a(r) = \infty.$$

We will prove that property (ii) is satisfied for function $V(t,x)$.

Indeed, for $(t,x) \in [0,\infty) \times \Omega_{B'}$, $t \neq t_k$, $k = 1,2,\dots$ and $\varepsilon > 0$, and for all $k \geq 1$ the inequality

$$\begin{aligned}
&V_k(t+\varepsilon, x(t+\varepsilon;t,x)) \\
&= \sup_{\eta\geq 0}\Big\{a(||x(t+\varepsilon+\eta;t+\varepsilon,x(t+\varepsilon;t,x))||)e^{c\eta}: \\
&\qquad\qquad t+\varepsilon+\eta \in (t_k,t_{k+1}]\Big\} \\
&= \sup_{\eta\geq 0}\Big\{a(||x(t+\varepsilon+\eta;t,x)||)e^{c\eta}: \ t+\varepsilon+\eta \in (t_k,t_{k+1}]\Big\} \\
&= e^{-c\varepsilon}\sup_{h\geq\varepsilon}\Big\{a(||x(t+h;t,x)||)e^{ch}: \ t+h \in (t_k,t_{k+1}]\Big\} \\
&\leq V_k(t,x)e^{-c\varepsilon}
\end{aligned}$$

holds.

The above inequality implies that

$$V(t+\varepsilon, x(t+\varepsilon;t,x)) \leq V(t,x)e^{-c\varepsilon}$$

and therefore

$$\limsup_{\varepsilon\to 0-}\frac{1}{\varepsilon}\Big\{V(t+\varepsilon,x(t+\varepsilon;t,x)) - V(t,x)\Big\} \leq -cV(t,x).$$

Finally we will prove that property (iii) is satisfied for function $V(t,x)$.

Indeed, for $x \in \Omega_{B'}$, $j = 1,2,\dots$ the inequality

$$V(t_j,x) = \sup_{k\geq j-1} V_k(t,x) \geq \sup_{k\geq j} V_k(t,x) = V(t_j+0, x+I_j(x))$$

holds. □

We will apply some of the obtained sufficient conditions to a system of impulsive differential equations.

Example 2.2.1. Consider the system of impulsive differential equations with fixed moments of impulses

$$x' = a(t)y - b(t)x(x^2+y^2), \tag{2.13}$$

$$y' = -a(t)x + b(t)y(x^2+y^2), \quad t \neq t_k, k = 1,2,\dots, \tag{2.14}$$

$$x(t_k+0)-x(t_k)=c_kx(t_k),\quad y(t_k+0)-y(t_k)=d_ky(t_k),\ k=1,2\ldots, \tag{2.15}$$

with initial condition

$$x(t_0)=x_0,\quad y(t_0)=y_0, \tag{2.16}$$

where $a(t),b(t)$ are continuous functions for $t\geq 0$, $b(t)\leq 0$, $-1<c_k\leq 0$, $-1<d_k\leq 0$.

Function $V(t,x,y)=x^2+y^2$ satisfies the conditions of Theorem 2.2.2.

Indeed, the derivative of function $V(t,x,y)$ along the trajectory of the solution of system (2.13)–(2.15) satisfies the inequality

$$D^+_{(2.13),(2.14),(2.15)}V(t,x,y)\leq 2b(t)(x^2+y^2)\leq 0.$$

At the moments of impulses from equalities (2.15) we obtain

$$V(t_k+0,x+c_kx,y+d_ky)=x^2(1+c_k)^2+y^2(1+d_k)^2\leq V(t_k,x,y).$$

According to the Theorem 2.2.2 the solutions of (2.13)–(2.15) are uniformly bounded.

If there exists a constant $\gamma>0$ such that $b(t)\leq-\gamma$, then according to Theorem 2.2.3 the solutions of (2.13)–(2.15) are uniform-ultimately bounded. □

2.3. Boundedness of the Solutions of Impulsive Equations with "Supremum"

We will use the piecewise continuous functions of Lyapunov, defined in the first section of this chapter, to study boundedness of the solutions of the impulsive differential equations with "supremum". We will define the derivative of the functions from the set W along the trajectory of the solution of the system of impulsive differential equations with "supremum".

Consider the initial value problem for the nonlinear impulsive differential equations with "supremum" and variable moments of impulses, i.e. the impulses occur on the sets σ_k, defined by the equality (30):

$$x'=f(t,x(t),\sup_{s\in[t-h,t]}x(s))\ \text{ for } t\geq t_0,\ t\neq\tau_k(x(t)), \tag{2.17}$$

$$x(t+0)-x(t-0)=I_k(x(t))\quad\text{for } t=\tau_k(x(t)), \tag{2.18}$$

$$x(t+t_0)=\varphi(t)\quad\text{for } t\in[-h,0], \tag{2.19}$$

where $x\in\mathbf{R}^n$, $f:[0,\infty)\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$, $\varphi:[-h,0]\to\mathbf{R}^n$, $I_k:\mathbf{R}^n\to\mathbf{R}^n$, $k=1,2,3,\ldots,$ $t_0\geq 0,$ $h=const>0$, $\tau_k:\mathbf{R}^n\to(0,\infty)$.

The solution of the initial value problem for the nonlinear impulsive differential equations with "supremum" (2.17), (2.18), (2.19) is denoted by $x(t;t_0,\varphi)$, and the maximal interval of existence of the solution is denoted by $J(t_0,\varphi)\subset[t_0-h,\infty)$.

Lets introduce the following conditions (H):

H1. Function $f\in C([0,\infty)\times\mathbf{R}^n\times\mathbf{R}^n,\mathbf{R}^n)$.

H2. Functions $I_k:\ \mathbf{R}^n\to\mathbf{R}^n$, $k=1,2,\ldots,$ are such that the inequality $\|x+I_k(x)\|<H$ holds if $\|x\|\leq H$ and $I_k(x)\neq 0$, where $H=const>0$.

H3. Functions $\tau_k \in C(\mathbf{R}^n, (0,\infty)), k = 1,2,\dots$ are such that $0 < \tau_1(x) < \tau_2(x) < \tau_3(x) < \dots$ for $x \in \mathbf{R}^n$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ uniformly in $x \in \mathbf{R}^n$.

H4. For $t_0 > 0$ and $\varphi \in PC([-h,0],\mathbf{R}^n)$ the solution of the initial value problem (2.17), (2.18), (2.19) exists on the interval $[t_0 - h, \infty)$.

H5. For $t > t_0$ the integral curve of each solution of the initial value problem (2.17), (2.18), (2.19) intersects any hypersurface $\tau_k, k = 1,2,\dots$ not more than once.

Definition 15. The solutions of the initial value problem for the nonlinear impulsive differential equation with "supremum" (2.17), (2.18), (2.19) are said to be *uniformly bounded*, if for every constant $\alpha > 0$ and for any $t_0 \geq 0$ there exists a constant $\beta = \beta(\alpha) > 0$ such that for each $\varphi \in PC([-h,0],\mathbf{R}^n)$: $sup\{\|\varphi(t)\| : t \in [-h,0]\} < \alpha$ the inequality $\|x(t;t_0,\varphi)\| < \beta$ holds for $t > t_0$.

Definition 16. The solutions of the initial value problem for the nonlinear impulsive differential equation with "supremum" (2.17), (2.18), (2.19) are said to be *uniform-ultimately bounded,* if there exists a constant $B > 0$ such that for every $\alpha > 0$ and $t_0 \geq 0$ there exits a constant $T = T(\alpha) > 0$ such that for each $\varphi \in PC([-h,0],\ R^n)$: $sup\{\|\varphi(t)\| : t \in [-h,0]\} < \alpha$ the inequality $||x(t;t_0,\varphi)|| < B$ holds for $t > t_0 + T$.

We define the derivative of function $V(t,x)$ from the set W along the trajectory of a solution of impulsive differential equations with "supremum" (2.17), (2.18) by the equality

$$D^-_{(2.17),(2.18)}V(t,\varphi(0)) = \limsup_{\varepsilon\to 0-}(1/\varepsilon)\{V(t+\varepsilon, x(t+\varepsilon;t,\varphi)) - V(t,\varphi(0))\} \tag{2.20}$$

for $(t,\varphi(0)) \notin \sigma_k,\ k = 1,2,\dots,$ where $\varphi \in PC([-h,0],\mathbf{R}^n)$.

We will obtain sufficient conditions for uniform boundedness of the solutions of the initial value problem (2.17), (2.18), (2.19).

Theorem 2.3.1. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*

2. *There exists a function* $V \in W$ *such that*

(i) $a(||x||) \leq V(t,x) \leq b(||x||)$ *for* $(t,x) \in [0,\infty) \times \mathbf{R}^n$, *where* $a \in \Delta, b \in CIP$;

(ii) *There exists a function* $p \in K$ *such that for each function* $\psi \in PC([-h,0],\ \mathbf{R}^n)$ *such that* $\|\psi(0)\| > H$ *and* $p(V(t,\psi(0))) > \sup\{V(t+s,\psi(s)) : s \in [-h,0]\}$ *for* $t > 0$, $t \neq \tau_k(\psi(0)), k = 1,2,3\dots$, *the inequality*

$$D^-_{(2.17),(2.18)}V(t,\varphi(0)) < 0$$

holds;

(iii) *The inequality* $V(t_k+0, x+I_k(x)) < V(t_k,x)$ *holds for* $t_k = \tau_k(x)$, $\|x\| > H$ *and* $I_k(x) \neq 0$.

Then the solutions of the initial value problem for the impulsive differential equations with "supremum" (2.17), (2.18), (2.19) *are uniformly bounded.*

Proof. Let $\alpha > 0$ be an arbitrary constant, $t_0 \geq 0$ be an arbitrary point and $\varphi \in PC([-h,0],\mathbf{R}^n)$ be such that $sup\{\|\varphi(s)\| : s \in [-h,0]\} < \alpha$.

We will assume that the integral curve $(t,x(t;t_0,\varphi))$ of the solution of the initial value problem (2.17), (2.18), (2.19) intersects each hyperface σ_{j_k}, $k = 1,2,3,\ldots$ at the points t_k, correspondingly, where $t_1 < t_2 < t_3 < \ldots$, i.e. $(t_k,x(t_k;t_0,\varphi)) \in \sigma_{j_k}$. From condition H3 follows that $\lim_{k\to\infty} t_k = \infty$.

Consider the following two cases:

Case 1. Let $\alpha > H$.

Introduce the notations $x(t) = x(t;t_0,\varphi)$ and $v(t) = V(t,x(t))$. It follows from the properties of the functions $a(t)$ and $b(t)$ that there exists a constant $\beta = \beta(\alpha) > 0$ such that $b(\alpha) < a(\beta)$, $\beta > \alpha$.

Let $t \in [t_0 - h,t_0]$. Condition (i), initial condition (2.19), and the choice of the function φ imply the inequalities

$$\begin{aligned} a(||x(t;t_0,\varphi)||) &= a(\|\varphi(t-t_0)\|) \leq V(t,\varphi(t-t_0)) \\ &\leq b(\|\varphi(t-t_0)\|) \leq b(\alpha) < a(\beta). \end{aligned}$$

Therefore

$$||x(t;t_0,\varphi)|| < \beta \quad \text{for} \quad t \in [t_0 - h,t_0]. \tag{2.21}$$

We will prove that

$$v(t) \leq a(\beta) \ \text{ for } \ t > t_0. \tag{2.22}$$

Suppose the contrary, i.e. there exists a point $t > t_0$ such that $v(t) > a(\beta)$. Introduce the notation

$$t^* = \inf\{t > t_0 : v(t) > a(\beta)\}.$$

We will prove that function $x(t)$ is continuous at the point t^*. If $t^* \neq t_k$ for every $k = 1,2,\ldots$, then the continuity follows from the definition of the solution of the impulsive equation.

If there exists a natural number $m: \ t^* = t_m$, then $I_m(x(t^*;t_0,\varphi) = 0$.

Indeed, if we assume the contrary, i.e. $I_m(x(t^*;t_0,\varphi) \neq 0$, then the following two cases can be distinguished:

Case 1.1. Let $||x(t_m)|| > H$. Then it follows from condition (iii) that

$$v(t_m+0) < v(t_m) \leq a(\beta). \tag{2.23}$$

Inequality (2.23) contradicts the choice of the point t^*.

Case 1.2. Let $||x(t_m)|| \leq H$. From condition H2 of Theorem 2.3.1 we have

$$||x(t_m) + I_m(x(t_m))|| < H. \tag{2.24}$$

From condition (i) we obtain that

$$v(t_m) \leq b(||x(t_m)||) \leq b(H) \leq b(\alpha) < a(\beta).$$

The choice of the point $t^* = t_m$ implies the inequality $v(t_m+0) \geq a(\beta)$. From condition (i) we conclude that

$$b(\|x(t_m) + I_m(x(t_m))\|) \geq v(t_m+0) \geq a(\beta) > b(\alpha). \tag{2.25}$$

Inequalities (2.25) imply

$$||x(t_m)+I_m(x(t_m)|| > \alpha \geq H.$$

The last inequaity contradicts (2.24).

Therefore, $t^* \neq t_k,\ k=1,2,3,\dots$ and the function $x(t)$ is continuous at point t^*.

From the continuity of functions $x(t)$ and $v(t)$ follows that function $v(t)$ possesses the following properties:

$(P1)$ $v(t^*) = a(\beta)$;
$(P2)$ $v(s) \leq a(\beta)$ for $t_0 - h \leq s < t^*$;
$(P3)$ there exists a sequence $\{\gamma_m\}$ such that $\gamma_m > t^*$ and $\lim_{m\to\infty}\gamma_m = t^*$ and $v(\gamma_m) > a(\beta)$.

From the properties of function $v(t)$ we obtain

$$v'(t^*) \geq 0. \tag{2.26}$$

Condition (i) and property $(P1)$ imply that

$$b(||x(t^*)||) \geq v(t^*) = a(\beta) > b(\alpha). \tag{2.27}$$

From the properties of function $b(t)$ and inequality (2.27) follows the validity of the inequality

$$||x(t^*)|| > \alpha \geq H. \tag{2.28}$$

According to properties $(P1)$ and $(P2)$ we get

$$V(t^*, x(t^*)) = v(t^*) = sup\{v(s) : s \in [t^* - h, t^*]\}.$$

We define function $\psi(t) \in PC([-h,0],\mathbf{R}^n)$ by the equality

$$\psi(t) = x(t^* + t; t_0, \varphi) \quad \text{for} \ \ t \in [-h, 0].$$

From inequality (2.28) follows that $||\psi(0)|| > H$ and $p(V(t,\psi(0))) > sup\{V(t+s,\psi(s)) : s \in [-h,0]\}$.

From condition (ii) we conclude that

$$D^-_{(2.17),(2.18)}V(t^*,\psi(0)) = D^-_{(2.17),(2.18)}V(t^*,x(t^*;t_0,\varphi)) = v'(t^*) < 0. \tag{2.29}$$

Inequality (2.29) contradicts inequality (2.26).

Therefore inequality (2.22) holds for $t > t_0$. From inequality (2.22) and condition (i) we obtain $\|x(t)\| \leq \beta$ for $t > t_0$.

Case 2. Let $\alpha < H$. We consider the following two cases:

Case 2.1. Let $||x(t;t_0,\varphi)|| < H$ for $t > t_0$. The solutions of the initial value problem (2.17), (2.18), (2.19) are uniformly bounded with a constant $\beta = H$.

Case 2.2. Let there exist a point $t > t_0$ such that $\|x(t;t_0,\varphi)\| \geq H$. Denote $\eta = inf\{t > t_0 : ||x(t;t_0,\varphi)|| \geq H\}$ and consider the solution $x(t;\eta,\psi)$ for $t > \eta$, where $\psi(t) = x(t+\eta;t_0,\varphi)$ for $t \in [-h,0]$, $sup\{||\psi(t)|| : t \in [-h,0]\} \leq H$ and $||x(t;t_0,\varphi)|| < H$ for $t \in [t_0,\eta)$. From case 1 follows that for $\alpha = H$ and $\varphi = \psi$ there exists a constant $\beta = \beta(H) > 0$ such that $||x(t;\eta,\psi)|| < \beta$ for $t > \eta$. Therefore, the solutions of the initial value problem (2.17), (2.18), (2.19) are uniformly bounded by a constant $H_1 = max(H,\beta(H))$. □

We will obtain sufficient conditions for uniform-ultimate boundedness of the solutions of the initial value problem (2.17), (2.18), (2.19).

Theorem 2.3.2. *Let the following conditions hold:*

1. *Conditions* (H) *are fulfilled.*

2. *There exists a function* $V \in W$ *such that the conditions (i) and* (iii) *of Theorem* 2.3.1 *are satisfied.*

3. *There exists a function* $p \in K$ *such that for each function* $\psi \in PC([-h,0],\mathbf{R}^n)$ *such that* $\|\psi(0)\| > H$ *and* $p(V(t,\psi(0))) > \sup\{V(t+s,\psi(s)) : s \in [-h,0]\}$ *for* $t > 0$, $t \neq \tau_k(\psi(0))$, $k = 1,2,3\ldots$, *the inequality*

$$D^-_{(2.17),(2.18)}V(t,\varphi(0)) < -c(||\varphi(0)||)$$

holds, where $c \in CIP$.

Then the solutions of the initial value problem for impulsive differential equations with "supremum" (2.17), (2.18), (2.19) *are uniform-ultimately bounded.*

Proof. From the properties of function $a(t)$ follows that a constant $B > 0$ exists such that $a(B) = b(H)$.

Let $\alpha > 0$ be an arbitrary number, $t_0 \geq 0$ be an arbitrary point, and function $\varphi \in C([-h,0],\mathbf{R}^n)$ be such that $sup\{\|\varphi(s)\| : s \in [-h,0]\} < \alpha$.

Consider the following two cases:

Case 1. Let $\alpha \geq H$. Having in mind the proof of Theorem 2.3.1, it is easy to verify that there exists a constant $\beta = \beta(\alpha) > 0$ such that

$$v(t) < a(\beta) \quad \text{for} \quad t \geq t_0. \tag{2.30}$$

The following two cases can be distinguished:

Case 1.1. Let $\beta \leq B$. Then inequality (2.30) and condition (i) imply that

$$a(||x(t;t_0,\varphi)||) \leq V(t,x(t;t_0,\varphi)) < a(\beta) \leq a(B), \ \ t \geq t_0. \tag{2.31}$$

We conclude from inequality (2.31) and the conditions of function $a(t)$ that

$$||x(t;t_0,\varphi)|| < B \quad \text{for} \quad t \geq t_0. \tag{2.32}$$

Inequality (2.32) proves that the solutions of the initial value problem for impulsive differential equations with "supremum" (2.17), (2.18), (2.19) are uniform-ultimately bounded.

Case 1.2. Let $\beta > B$. Then $a(\beta) > a(B)$. Consider the interval $\Delta = [a(B), a(\beta)]$. We will introduce the notation

$$A = inf\{p(s) - s : s \in \Delta\} > 0. \tag{2.33}$$

Let N be the smallest integer such that $N \geq (a(\beta) - a(B))/A$. Define points $\xi_j = t_0 + j(h + 2A/c(H))$, $j = 0,1,\ldots,N$. We will prove the validity of the inequality

$$v(t) < a(B) + (N-j) \quad \text{for } t > \xi_j, \ \ j = 0,1,\ldots,N. \tag{2.34}$$

Let $j=0$. Then $\xi_0=t_0$. The choice of N as well as inequality (2.30) imply that

$$v(t)<a(\beta)\leq a(B)+NA=a(B)+(N-0)A \quad \text{for } \ t>t_0. \tag{2.35}$$

Therefore, inequality (2.34) is fulfilled for $j=0$.

Suppose that inequality (2.34) holds for $t>\xi_j,\ j=0,1,\ldots,l\ \ (l<N)$. We will prove that inequality (2.34) is also fulfilled for $t>\xi_{j+1}$.

First, we will prove that there exists a point $\eta\in[\xi_l+h,\xi_{l+1}]$ such that the inequality

$$v(\eta)<a(B)+(N-l-1)A \tag{2.36}$$

holds.

Suppose the contrary, i.e. that for $t\in[\xi_l+h,\xi_{l+1}]$ the inequalities

$$a(B)+(N-l-1)A\leq v(t)<a(B)+(N-l)A \tag{2.37}$$

hold.

From condition (i), inequalities (2.37), and the choice of B we obtain that

$$b(||x(t;t_0,\varphi||))\geq v(t)>a(B)+(N-l-1)A\geq a(B)=b(H). \tag{2.38}$$

From inequality (2.38) follows that

$$||x(t;t_0,\varphi)||>H \quad \text{for } \ t\in[\xi_l+h,\xi_{l+1}]. \tag{2.39}$$

The properties of function $p(t)$, the choice of number A, inequality (2.39), and the inequalities $a(B)<V(t)<a(\beta)$ imply that

$$\begin{aligned} p(v(t)) &\geq v(t)+A>a(B)+(N-l)A \\ &\geq sup\{v(t+s):s\in[-h,0]\} \quad \text{for } \ t\in[\xi_l+h,\xi_{l+1}]. \end{aligned} \tag{2.40}$$

From condition 3 of Theorem 2.3.2, inequalities (2.37) and (2.40), and the properties of the functions of class K follows that

$$v'(t)\leq -c(\|x(t;t_0,\varphi)\|)<-c(H) \quad \text{for } \ t\in[\xi_l+h,\xi_{l+1}]. \tag{2.41}$$

Choose a point $\zeta\in[\xi_l+h+\beta,\xi_{l+1}]$, where $\beta=(\xi_{l+1}-\xi_l-h)/2=A/c(H)$. We have from (2.37), (2.38), the condition (ii) and the inequality $\zeta-\xi_l-h>\beta$ that

$$\begin{aligned} v(\xi) &= v'(\gamma)(\zeta-\xi_l-h)+v(\xi_l+h) \\ &< -c(H)(\zeta-\xi_l-h)+v(\xi_l+h)\leq a(B)+(N-l-1)A, \end{aligned} \tag{2.42}$$

where $\gamma\in(\xi_l+h,\zeta)$.

Inequality (2.42) contradicts (2.37).

Therefore there exists a point $\eta\in[\xi_l+h,\xi_{l+1}]$ such that inequality (2.36) is true.

We will prove that (2.36) is valid for $t>\eta$. Suppose the contrary and denote

$$t^{**}=inf\{t:t\geq\eta,v(t)\geq a(B)+(N-l-1)A\}.$$

Having in mind the choice of points t^* and t^{**} and the continuity of $v(t)$, we conclude that function $v(t)$ possesses the following properties:

$(P1)$ $\quad v(t^{**}) = a(B) + (N-l-1)A;$

$(P2)$ $\quad v(s) < a(B) + (N-l-1)A \quad \text{for} \quad \eta \le s < t^{**};$

$(P3)$ there exists a sequence $\{\eta_m\}$ such that $\eta_m > t^{**}$ and $\lim_{m\to\infty} \eta_m = t^{**}$, and $v(\eta_m) \ge a(B) + (N-l-1)A$.

It follows from the above properties of the function $v(t)$ that

$$v'(t^{**}) \ge 0. \tag{2.43}$$

Conditions (i), the choice of point t^{**}, and the choice of the constant β imply that

$$b(||x(t^{**};t_0,\varphi)||) \ge v(t^{**}) = a(B) + (N-l-1)A > a(\beta) = b(H). \tag{2.44}$$

Inequality (2.44) and the monotonicity of $b(t)$ imply that

$$||x(t^{**};t_0,\varphi)|| > H. \tag{2.45}$$

Then from properties (P1) and (P2) of function $v(t)$ and the properties of $p(t)$ we obtain that

$$p(v(t^{**})) > v(t^{**}) = a(B) + (N-l-1)A > v(s) \quad \text{for } s \in [\eta, t^{**}]. \tag{2.46}$$

Inequalities (2.45), (2.46) and condition 3 lead to

$$v'(||x(t^{**};t_0,\varphi)||) \le -c(||x(t^{**};t_0,\varphi)||) < 0. \tag{2.47}$$

Inequality (2.47) contradicts (2.43).

The obtained contradiction proves that inequality (2.36) is valid for $t > \eta$ and therefore for $t > \xi_{l+1}$.

Thus, the validity of inequality (2.34) has been proved.

Set $T = \tau_N - \tau_0 = N(h + 2A/c(H)) > 0$. Note that T depends only on α.

We have from the condition (i) and inequality (2.34) for $j = N$ that

$$a(||x(t;t_0,\varphi)||) \le v(t) < a(B) \quad \text{for} \quad t > \xi_N = t_0 + T. \tag{2.48}$$

Therefore, $\|x(t;t_0,\varphi)\| < B$ for $t > t_0 + T$ which proves that the solutions of the initial value problem for impulsive differential equations with "supremum" (2.17), (2.18), (2.19) are uniform-ultimately bounded.

Case 2. Let $\alpha < H$. The proof of this case is analogous to the proof of case 2 of Theorem 2.3.1. □

As a consequence of Theorem 2.3.2 we obtain the following result:

Theorem 2.3.3. *Let the following conditions hold:*

1. *Conditions* (H) *are fulfilled.*

2. *There exists a function* $V \in W$ *that satisfies conditions* (i) *and* (iii) *of Theorem* 2.3.1.

3. *There exists a function* $p \in K$ *such that for each function* $\psi \in PC([-h,0], \mathbf{R}^n)$ *such that* $\|\psi(0)\| > H$ *and* $p(V(t,\psi(0))) > \sup\{V(t+s,\psi(s)) : s \in [-h,0]\}$ *for* $t > 0,\ t \ne$

$\tau_k(\psi(0))$, $k = 1,2,3\dots$, *the inequality* $D^-_{(2.17),(2.18)}V(t,\psi(t)) < M - d(\|\psi(0)\|)$ *is valid, where* $d \in K$, $Q = const > 0$, $M = const > 0$, $\lim_{s\to\infty} d(s) = \infty$.

Then the solutions of the initial value problem for impulsive differential equations with "supremum" (2.17), (2.18), (2.19) *are uniform-ultimately bounded.*

Proof. From the properties of function $d(s)$ follows that there exists a constant $C = C(M) > 0$ such that $d(s) - M > 0$ for $s > C$.

Define function $c : [0,\infty) \to [0,\infty)$ by

$$c(s) = \begin{cases} d(s) - M & \text{for } s > C \\ \frac{(d(s)-M)s}{C} & \text{for } 0 \le s \le C \end{cases}.$$

Function $c(s) \in K$. Then the conditions of Theorem 2.3.2 are fulfilled with a constant $H = max(C,Q)$ and therefore the solutions of the initial value problem for impulsive differential equations with "supremum" (2.17), (2.18), (2.19) are uniform-ultimately bounded. □

Remark 6. We will note that in the case $I_k(x) = 0$ for $k = 1,2,3,\dots$ the initial value problem (2.17), (2.18), (2.19) reduces to the initial value problem for differential equations with "maxima"

$$x'(t) = f(t, x(t), \max_{s\in[t-h,t]} x(s)) \quad \text{for } t \ge 0,$$

$$x(t+t_0) = \varphi(t) \quad \text{for } t \in [-h,0]$$

and Theorem 2.3.1, Theorem 2.3.2, and Theorem 2.3.3 give sufficient conditions for boundedness of the solutions of differential equations with "maxima".

Example 2.3.1. Consider the initial value problem for the following scalar impulsive differential equation with "supremum"

$$x(t)x'(t) = -f(t)x^2(t) + g(t)x(t) \sup_{s\in[t-h,t]} x(s) + h(t) \quad \text{for } t \ge t_0,\ t \ne t_k, \tag{2.49}$$

$$x(t_k+0) = (1+c_k)x(t_k), \quad k = 1,2,\dots, \tag{2.50}$$

$$x(t+t_0) = \varphi(t) \quad \text{for } t \in [-h,0], \tag{2.51}$$

where $x \in \mathbf{R}$, $0 < t_1 < t_2 < \dots$, $c_k = const$, $k = 1,2,3,\dots$.

Let the following conditions hold:

A1. Function $f \in C([0,\infty),(0,\infty))$.

A2. Function $g \in C([0,\infty),\mathbf{R})$ and there exist constants $L > 0$ and $q > 1$ such that $f(t) - L \ge q|g(t)|$ for $t \ge 0$.

A3. Function $h \in C([0,\infty),\mathbf{R})$ and there exists a constant $M > 0$ such that $|h(t)| \le M$ for $t \ge 0$.

A4. The limit $\lim_{k\to\infty} t_k = \infty$.

A5. The inequality $-2 < c_k < 0$, $k = 1,2,3,\dots$ holds.

A6. Function $\varphi \in C([-h,0],\mathbf{R})$.

Consider function $V(t,x) = x^2/2$ which is from set W.

Define functions $a(s) = s^2/4$, $b(s) = s^2$, $p(s) = q^2$, and $d(s) = Ls^2$. It is easy to verify that condition (i) of Theorem 2.3.1 is satisfied.

Let $t \geq 0$ be an arbitrary point and function $\psi \in C([-h,0],\mathbf{R})$ be such that $q|\psi(0)| > sup\{|\psi(s)| : s \in [-h,0]\}$ and $|\psi(0)| > H$. Then for $s \in [-h,0]$ the inequality $p(V(t,\psi(0)) = q^2 V(t,\psi(0)) = q^2\psi^2(0)/2 > \psi^2(s)/2 = V(s,\psi(s))$ holds.

We obtain for the derivative along the trajectory of the solution of (2.49), (2.50) the following inequalities

$$\begin{aligned} D^-_{(2.49),(2.50)}V(t,\psi(0)) &= -f(t)\psi^2(0) + g(t)\psi(0)\sup_{s\in[-h,0]}\psi(s) + h(t) \\ &\leq -f(t)\psi^2(0) + |g(t)|.|\psi(0)|.|\sup_{s\in[-h,0]}\psi(s)| + |h(t)| \\ &\leq -f(t)\psi^2(0) + q|g(t)|\psi^2(0) + |h(t)| \\ &\leq M - L\psi^2(0) = M - d(|\psi(0)|). \end{aligned} \tag{2.52}$$

Inequality (2.52) shows the validity of condition 3 of Theorem 2.3.3. From condition A5 follows the validity of the inequality

$$V(t_k+0, x+c_k x) = \frac{x^2(1+c_k)^2}{2} < \frac{x^2}{2}, \;\; x \neq 0.$$

Let $|x| \leq H$. Then the following inequalities are fulfilled

$$|x + c_k x| = |x|.|1+c_k| < H.|1+c_k| < H.$$

Therefore, condition H2 is fulfilled.

We conclude from Theorem 2.3.3 that if conditions (A) are fulfilled then the solutions of initial value problem (2.49), (2.50), (2.51) are uniform-ultimately bounded.

It follows from inequality (2.52) that if $H > (M/L)^{1/2}$ then all conditions of Theorem 2.3.1 are satisfied and therefore the solutions of initial value problem (2.49), (2.50), (2.51) are uniformly bounded.

2.4. Boundedness of the Solutions of Impulsive Hybrid Equations

Let points $t_k, (k = 0,1,2,\ldots,)$ be fixed such that $t_k < t_{k+1}$ and $\lim_{k\to\infty} t_k = \infty$. We will asume that $t_0 \geq 0$.

Consider the initial value problem for the system of nonlinear impulsive hybrid equations with fixed moments of impulses

$$x' = f(t, x(t), \lambda_k(x(t_k))) \quad \text{for} \quad t \in (t_k, t_{k+1}),\; k = 0,1,2,\ldots, \tag{2.53}$$

$$x(t_k+0) = x(t_k) + I_k(x(t_k)),\; k = 1,2,\ldots, \tag{2.54}$$

$$x(t_0) = x_0, \tag{2.55}$$

where $x \in \mathbf{R}^n$, $f : [0,\infty) \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $(k = 1,2,3,\ldots)$, $\lambda_k : \mathbf{R}^n \to \mathbf{R}^m$, $(k = 0,1,2,3,\ldots)$.

We denote by $x(t;t_0,x_0)$ the solution of the initial value problem for the system of impulsive hybrid equations (2.53), (2.54), (2.55), and by $J(t_0,x_0)$ the maximal interval of the type $[t_0,\beta)$, on which the solution $x(t;t_0,x_0)$ is defined.

We will note that the solution of the initial value problem (2.53), (2.54), (2.55) depends not only on the initial point (t_0,x_0), but on $\lambda_0(x_0)$. Indeed, let $\tau_0 :\ t_0 < \tau_0 < t_1$. If problem (2.53), (2.54), (2.55) has unique solution, then it is naturally to assume that $x(t;t_0,x_0) = x(t;\tau_0,\kappa_0)$ for $t \geq \tau_0$, where $\kappa_0 = x(\tau_0;t_0,x_0)$. The solution $x(t;\tau_0,\kappa_0)$ depends not only on the initial point (τ_0,κ_0), but on $\lambda_0(x_0)$. We denote the solution of (2.53), (2.54), (2.55) by $x(t;t_0,x_0,\lambda_0(x_0))$.

We will say that conditions (H) are satisfied if:

H1. Function $f \in C([0,\infty)\times\mathbf{R}^n\times\mathbf{R}^m,\mathbf{R}^n)$ and $\lambda_k \in C(\mathbf{R}^n,\mathbf{R}^m)$, $(k = 0,1,2,\dots)$.

H2. Functions $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $k = 1,2,\dots$ are such that for $\|x\| \leq H$ the inequality $\|x+I_k(x)\| < H$ holds and $I_k(x) \neq 0$, where $H = const > 0$.

Definition 17. The solutions of the initial value problem for the system of nonlinear impulsive hybrid equations (2.53), (2.54), (2.55) are called *uniformly bounded* , if for any number $\alpha > 0$ and for any point $t_0 \geq 0$ there exists a number $\beta = \beta(\alpha) > 0$ such that for $x_0 \in \mathbf{R}^n : \|x_0\| < \alpha$ and for $t > t_0$ the inequality $\|x(t;t_0,x_0,\lambda(x_0))\| < \beta$ holds.

Definition 18. The solutions of the initial value problem for the system of nonlinear impulsive hybrid equations (2.53), (2.54), (2.55) are called *uniform ultimately bounded*, if there exists a constant $B > 0$ such that for any number $\alpha > 0$ and for any point $t_0 \geq 0$ there exists a constant $T = T(\alpha) > 0$ such that for all points $x_0 \in \mathbf{R}^n\ :\ \|x_0\| < \alpha$ the inequality $\|x(t;t_0,x_0,\lambda(x_0))\| < B$ holds for $t > t_0+T$.

Let function $V(t,x)$ be from the set W. We define a derivative of functions $V(t,x,\lambda_k(y))$ along the trajectory of a solution of the system of nonlinear impulsive hybrid equations (2.53), (2.54) by the equality

$$D^-_{(2.53),(2.54)}V(t,x,\lambda_k(y))$$
$$= \limsup_{\varepsilon\to 0-}(1/\varepsilon)\{V(t+\varepsilon,x(t+\varepsilon;t,x,\lambda_k(y)))-V(t,x)\}, \qquad (2.56)$$

for $t \in (t_k,t_{k+1}), (k = 0,1,2,\dots)$, $x,y \in \mathbf{R}^n$.

We will obtain sufficient conditions for uniform boundedness of the solutions of the initial value problem (2.53), (2.54), (2.55).

Theorem 2.4.1. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*
2. *There exists a function* $V \in W$ *such that*

$$a(||x||) \leq V(t,x) \leq b(||x||) \quad \textit{for } t \geq 0,\ ||x|| > H,$$

where $a \in \Delta, b \in CIP$.

3. *There exists a function* $p \in K_1$ *such that for all functions* $x \in PC([0,\infty),\mathbf{R}^n)$ *such that* $||x(t)|| > H$ *and* $p(V(t,x(t))) > V(t_k,x(t_k))$ *for* $t \in (t_k,t_{k+1})$ *the inequality*

$$D^-_{(2.53),(2.54)}V(t,x(t),\lambda_k(x(t_k))) \leq G(t)F(V(t,x(t)),\ t \in (t_k,t_{k+1})$$

holds, where the functions $G,F:[0,\infty)\to[0,\infty)$ *are integrable and* $H>0$ *is a constant.*

4. $p(V(t_k,x+I_k(x)))\le V(t_k,x)$ *for* $k=1,2,3,\ldots$, $\|x\|>H$ *and* $I_k(x)\ne 0$.

5. *There exists a constant* $A>0$ *such that*

$$\int_{t_{k-1}}^{t_k} G(s)\,ds<A \quad \textit{and} \quad \int_{\mu}^{p(\mu)}\frac{ds}{F(s)}\ge A \ \textit{ for } \ \mu>0 \textit{ and } \ k=1,2,3\ldots.$$

Then the solutions of the initial value problem for the system of nonlinear impulsive hybrid equations (2.53), (2.54), (2.55) *are uniformly bounded.*

Proof. Let $\alpha>0$ be an arbitrary number, $t_0\ge 0$ be an arbitrary point and $x_0\in\mathbf{R}^n$: $\|x_0\|<\alpha$. We assume that $t_0<t_1$. Denote $v(t)=V(t,x(t;t_0,x_0,\lambda_0(x_0)))$.

Choose the constant $\beta=\beta(\alpha)>0$ such that

$$p(b(\alpha))<\alpha(\beta). \tag{2.57}$$

Then $b(\alpha)<\alpha(\beta)$.

Consider the following two cases:

Case 1. Let $\alpha>H$ and $H<\|x_0\|<\alpha$. We will prove that

$$v(t)<p(b(\alpha)) \quad \text{for} \ \ t>t_0. \tag{2.58}$$

If inequality (2.58) is not true for $t\in(t_0,t_1]$, then from the continuity of function $v(t)$ and inequality $v(t_0)=V(t_0,x_0)\le b(\|x_0\|)<b(\alpha)<p(b(\alpha))$ it follows that there exists a point $\xi\in(t_0,t_1)$ such that

$$\begin{gathered} v(\xi)=p(b(\alpha)),\\ v(t)<p(b(\alpha)) \quad \text{for} \quad t\in[t_0,\xi),\\ v'(\xi)\ge 0. \end{gathered} \tag{2.59}$$

From inequality $v(t_0)=V(t_0,x_0)<b(||x_0||)<b(\alpha)$ follows that there exists a point $\eta\in(t_0,\xi)$ such that

$$\begin{gathered} v(\eta)=b(\alpha),\\ v(t)>b(\alpha) \quad \text{for} \quad t\in(\eta,\xi],\\ v'(\eta)\ge 0. \end{gathered} \tag{2.60}$$

Therefore, for all $t\in[\eta,\xi]$ from the inequality (2.60) we obtain

$$p(V(t,x(t;t_0,x_0,\lambda_0(x_0))))=p(v(t))\ge p(b(\alpha))>V(t_0,x_0)$$

and

$$b(\|x(t;t_0,x_0,\lambda_0(x_0))\|)\ge V(t,x(t;t_0,x_0,\lambda_0(x_0)))=v(t)\ge b(\alpha),$$

i.e. $\|x(t;t_0,x_0,\lambda_0(x_0))\|\ge\alpha>H$.

According to condition 3 of Theorem 2.4.1 follows the validity of the inequality

$$\begin{aligned} &D^-_{(2.53),(2.54)}V(t,x(t;t_0,x_0,\lambda_0(x_0)),\lambda_0(x_0))\\ &\le G(t)F(V(t,x(t;t_0,x_0,\lambda_0(x_0))))\\ &=G(t)F(v(t)) \quad \text{for} \quad t\in[\eta,\xi]. \end{aligned}$$

From the continuity of the function $x(t;t_0,x_0,\lambda_0(x_0))$ on the interval $[\eta,\xi]\subset(t_0,t_1)$ it follows that

$$v'(t)=D^-_{(2.53),(2.54)}V(t,x(t;t_0,x_0,\lambda_0(x_0)),\lambda_0(x_0)),$$

i.e. the inequality

$$v'(t)\le G(t)F(v(t)) \tag{2.61}$$

holds.

We integrate inequality (2.61) on interval (η,ξ), we use condition 5 of Theorem 2.4.1 and we obtain

$$\int_\eta^\xi \frac{v'(t)}{F(v(t))}dt=\int_{v(\eta)}^{v(\xi)}\frac{du}{F(u)}\le\int_\eta^\xi G(s)ds\le\int_{t_0}^{t_1}G(s)ds<A. \tag{2.62}$$

From the choice of the points ξ and η and the condition 5 we obtain the inequality

$$\int_{v(\eta)}^{v(\xi)}\frac{du}{F(u)}=\int_{b(\alpha)}^{p(b(\alpha))}\frac{du}{F(u)}\ge A. \tag{2.63}$$

Inequality (2.63) contradicts inequality (2.62). Therefore inequality (2.58) holds for $t\in(t_0,t_1]$.

From condition 4 and inequality (2.58) for $t=t_1$ follows that

$$p(V(t_1+0,x(t_1+0;t_0,x_0,\lambda_0(x_0)))\le V(t_1,x(t_1;t_0,x_0,\lambda_0(x_0)))<p(b(\alpha)),$$

i.e.

$$V(t_1+0,x(t_1+0;t_0,x_0,\lambda_0(x_0)))<b(\alpha). \tag{2.64}$$

We will prove that inequality (2.58) holds for $t\in(t_1,t_2]$.

Assume the contrary. Then there exists a point $\xi_1\in(t_1,t_2)$ such that

$$V(\xi_1,x(\xi_1;t_0,x_0,\lambda_0(x_0)))>p(b(\alpha))>b(\alpha)>V(t_1+0,x(t_1+0;t_0,x_0,\lambda_0(x_0)). \tag{2.65}$$

Inequality (2.65) implies that there exists a point $\eta_1\in(t_1,\xi_1)$ such that

$$v(\eta_1)=p(b(\alpha)),$$

$$v(t)<p(b(\alpha))\quad\text{for}\quad t\in(t_1,\eta_1].$$

From the choice of point η_1 and the properties of function $p(s)$ follows that there exists a point $\eta_2\in(t_1,\eta_1)$ such that

$$v(\eta_2)=b(\alpha),$$

$$v(t)>b(\alpha)\quad\text{for}\quad t\in(\eta_2,\eta_1].$$

Therefore for $t\in[\eta_2,\eta_1)$ we obtain

$$p(v(t))\ge p(b(\alpha))>v(t_1)$$

and

$$p(\|x(t;t_0,x_0,\lambda_0(x_0))\|)\ge v(t)\ge b(\alpha),$$

i.e. $\|x(t;t_0,x_0,\lambda_0(x_0))\| \geq \alpha > H$.

According to condition 3 we conclude that for $t \in [\eta_2,\eta_1]$ the inequality

$$D^-_{(2.53),(2.54)}V(t,x(t;t_1,x_1,\lambda_1(x_1)),\lambda_1(x_1))$$
$$\leq G(t)F(V(t,x(t;t_1,x_1,\lambda_1(x_1))))$$

holds, where $x_1 = x(t_1;t_0,x_0,\lambda_0(x_0))$.

As in the proof of inequalities (2.62) and (2.63) we obtain a contradiction.

Therefore inequality (2.58) holds for $t \in (t_1,t_2]$.

Then

$$\begin{aligned} p(V(t_2+0,x(t_2+0;t_0,x_0,\lambda_0(x_0))) &\leq V(t_2,x(t_2;t_0,x_0,\lambda_0(x_0))) \\ &< p(b(\alpha)), \end{aligned}$$

i.e.

$$V(t_2+0,x(t_2+0;t_0,x_0,\lambda_0(x_0))) < b(\alpha).$$

Using induction, we can prove the validity of the following inequalities

$$v(t) < p(b(\alpha)) \quad \text{for} \quad t \in (t_k,t_{k+1}], \tag{2.66}$$

$$v(t_k+0) < b(\alpha). \tag{2.67}$$

From inequalities (2.66), (2.67), and $p(s) > s$ follows the validity of inequality (2.58) for $t > t_0$.

The condition (i) and the inequalities (2.56), (2.58) imply that for $t \geq t_0$ the inequalities

$$a(\|x(t;t_0,x_0,\lambda_0(x_0))\|) \leq v(t) < p(b(\alpha)) < a(\beta)$$

hold.

Therefore

$$\|x(t;t_0,x_0,\lambda_0(x_0))\| < \beta \quad \text{for} \quad t \geq t_0.$$

Case 2. Let $\alpha \leq H$. Consider the following two cases:

Case 2.1. Let $\|x(t;t_0,x_0,\lambda_0(x_0))\| < H$ for $t > t_0$. Then the solutions of the initial value problem for the system of nonlinear impulsive hybrid equations (2.53), (2.54), (2.55) are uniformly bounded with a constant $\beta = H$.

Case 2.2. Let there exist a point $t > t_0$ such that $\|x(t;t_0,x_0,\lambda_0(x_0))\| \geq H$ holds. We denote $\eta = inf\{t > t_0 : \|x(t;t_0,x_0,\lambda_0(x_0))\| \geq H\}$. Let $\eta \in (t_j,t_{j+1}]$. Consider the solution $x(t;\eta,y,\lambda_j(y))$ for $t > \eta$, where $y = x(\eta;t_0,x_0,\lambda_0(x_0))$, $\|y\| < H$ and $\|x(t;t_0,x_0,\lambda_0(x_0))\| < H$ for $t \in [t_0,\eta)$. From case 1 follows that for $\alpha = H$ there exists a constant $\beta = \beta(H) > 0$ such that $\|x(t;\eta,y,\lambda_j(y))\| < \beta$ for $t > \eta$. Therefore the solutions of the initial value problem for the system of nonlinear impulsive hybrid equations (2.53), (2.54), (2.55) are uniformly bounded by the constant $H_1 = max(H,\beta(H))$. □

We will obtain sufficient conditions for uniform-ultimate boundedness of the solutions of the initial value problem for the system of impulsive hybrid equations (2.53), (2.54), (2.55).

Theorem 2.4.2. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*

2. *There exists a function* $V \in W$ *such that*

$$a(||x||) \le V(t,x) \le b(||x||) \quad \text{for} \quad ||x|| > H,$$

where $a \in \Delta$, $b \in CIP$.

3. *There exists a function* $p \in K_1$ *such that for all functions* $x \in PC([0,\infty),\mathbf{R}^n)$*, such that* $||x(t)|| > H$ *and* $p(V(t,x(t)) > V(t_k,x(t_k))$ *for* $t \in (t_k,t_{k+1})$ *the inequality*

$$D^-_{(2.53),(2.54)}V(t,x(t),\lambda_k(x(t_k))) \le G(t)F(V(t,x(t))), \;\; t \in (t_k,t_{k+1})$$

holds, where $G : [0,\infty) \to [0,\infty)$ *is an integrable function,* $F \in C([0,\infty),[0,\infty))$ *is a nondecreasing function and* $F(0) = 0, F(s) > 0$ *for* $s > 0$.

4. *Inequality* $p(V(t_k,x+I_k(x))) \le V(t_k,x)$ *holds for* $k = 1,2,3,\ldots$, $\|x\| > H$ *and* $I_k(x) \ne 0$.

5. *Constants* $\rho_1 \ge \rho_2 > 0$ *and* $A > 0$ *exist such that for* $\mu > 0$, $\rho_1 \ge t_k - t_{k-1} \ge \rho_2$ *and* $k = 1,2,3\ldots$ *the inequality*

$$\int_{\mu}^{p(\mu)} \frac{ds}{F(s)} - \int_{t_{k-1}}^{t_k} G(s)ds \ge A$$

holds.

The solutions of the initial value problem for the system of impulsive hybrid equations (2.53), (2.54), (2.55) *are uniform-ultimately bounded.*

Proof. From the properties of the functions a,b,p it follows that there exists a number $B = B(H) > 0$ such that

$$p(b(H)) < a(B). \tag{2.68}$$

Let $\alpha > 0$ be an arbitrary number, $t_0 \ge 0$ be an arbitrary point and $x_0 \in \mathbf{R}^n: \;\; \|x_0\| < \alpha$. We will assume that $t_0 < t_1$.

We denote $v(t) = V(t,x(t;t_0,x_0,\lambda_0(x_0)))$, and let N be the least natural number such that

$$p(b(\alpha)) \le p(a(B)) + NAF(p^{-1}(a(B))),$$

where $p^{-1}(s)$ is the inverse function of the function $p(s)$.

From the properties of function $p(s)$ and condition 4 of Theorem 2.4.2 we obtain the inequalities

$$v(t_k+0) < p((t_k+0)) \le v(t_k), \quad k = 1,2,3,\ldots.$$

We introduce the notation

$$L_k = \sup\{v(t) : t \in (t_k,t_{k+1}]\}, \;\; k = 0,1,\ldots.$$

We consider the following two cases:

Case 1. Let for any natural number k there exist a point $\xi_k \in (t_k,t_{k+1})$ such that $v(\xi_k) = L_k$.

We will prove that there exists an integer $i : 1 \leq i \leq N$ such that

$$p(v(\xi_i)) \leq a(B). \tag{2.69}$$

If inequality (2.69) is not true, then for all integers $i : 1 \leq i \leq N$ the inequality

$$p(v(\xi_i)) > a(B) \tag{2.70}$$

holds.

With the help of mathematical induction we will prove that

$$v(\xi_i) \leq p(b(\alpha)) - iAF(p^{-1}(a(B))). \tag{2.71}$$

We assume that inequality (2.71) holds for an integer $i : 1 \leq i < N$. We will prove that this inequality is true for $i+1$.

Initially we will prove that

$$v(\xi_{i+1}) \leq v(\xi_i), \quad i = 0, 1, 2, \dots \tag{2.72}$$

where $v(\xi_0) = p(b(\alpha))$. Indeed, from the choice of the points ξ_i and condition 4 of Theorem 2.4.2 it follows that

$$v(t) \leq v(\xi_i), \quad t \in (t_i, t_{i+1}]$$

and

$$v(t_i + 0) < p(v(t_i + 0)) \leq v(t_i) \leq v(\xi_{i-1}).$$

According to inequality (2.72) and the properties of the function $p(s)$ the following two cases are possible:

Case 1.1. Let $a(B) < p(v(\xi_{i+1})) < v(\xi_i)$. From condition 5 of Theorem 2.4.2, inequality (2.72), and the properties of function $F(u)$ follows that

$$A \leq \int_{v(\xi_{i+1})}^{p(v(\xi_{i+1}))} \frac{ds}{F(s)} \leq \int_{v(\xi_{i+1})}^{p(v(\xi_{i+1}))} \frac{ds}{F(p^{-1}(a(B)))}$$
$$\leq \frac{p(v(\xi_{i+1})) - v(\xi_{i+1})}{F(p^{-1}(a(B)))}. \tag{2.73}$$

From inequality (2.73) and the inductive assumption we obtain

$$\begin{aligned} v(\xi_{i+1}) &\leq p(v(\xi_{i+1})) - AF(p^{-1}(a(B))) \\ &\leq v(\xi_i) - AF(p^{-1}(a(B))) \end{aligned}$$
$$\leq p(b(\alpha)) - (i+1)AF(p^{-1}(a(B))). \tag{2.74}$$

From inequality (2.74) follows that inequality (2.71) holds for $i = 1, 2, \dots.$

Case 1.2. Let $v(\xi_i) < p(v(\xi_{i+1})) \leq p(v(\xi_i))$.

From condition 4 of Theorem 2.4.2 we obtain

$$p(v(t_{i+1} + 0)) \leq v(t_{i+1}) \leq v(\xi_i).$$

Therefore there exists a point $\tau \in (t_{i+1}, \xi_{i+1})$ such that

$$p(v(\tau)) = v(\xi_i)$$

and

$$p(v(t)) > v(\xi_i) \geq v(t_{i+1}) \quad \text{for} \quad t \in (\tau, \xi_{i+1}).$$

From condition 3 of Theorem 2.4.2 we obtain that on $(\tau, \xi_{i+1}) \subset (t_{i+1}, t_{i+2})$ the inequality

$$\begin{aligned} &D^-_{(2.53),(2.54)} V(t, x(t; t_{i+1}, x_{i+1}, \lambda_{i+1}(x_{i+1})), \lambda_{i+1}(x_{i+1})) \\ &\leq G(t) F(V(t, x(t; t_{i+1}, x_{i+1}, \lambda_{i+1}(x_{i+1})))) \end{aligned}$$

holds, where $x_{i+1} = x(t_{i+1}; t_0, x_0, \lambda_0(x_0))$.

Therefore the inequality

$$\begin{aligned} \int_{v(\tau)}^{v(\xi_{i+1})} \frac{ds}{F(s)} &\leq \int_{\tau}^{\xi_{i+1}} G(s)ds \leq \int_{t_i}^{t_{i+1}} G(s)ds \\ &\leq \int_{v(\tau)}^{p(v(\tau))} \frac{ds}{F(s)} - A = \int_{v(\tau)}^{v(\xi_i)} \frac{ds}{F(s)} - A \end{aligned} \tag{2.75}$$

holds.

Then

$$\int_{v(\xi_{i+1})}^{v(\xi_i)} \frac{ds}{F(s)} \geq A,$$

that implies

$$\begin{aligned} v(\xi_{i+1}) &\leq v(\xi_i) - AF(p^{-1}(a(B))) \\ &\leq p(b(\alpha)) - (i+1)AF(p^{-1}(a(B))). \end{aligned} \tag{2.76}$$

Case 2. There exists a natural number k such that $v(t_k) = L_k$. The proof of case 2 is analogous to the proof of case 1.

Therefore we proved the validity of inequality (2.71).

We set $i = N$ in inequality (2.71). Then from the choice of number N follows that

$$v(\xi_N) \leq p(b(\alpha)) - NAF(p^{-1}(a(B))) \leq p(a(B)). \tag{2.77}$$

Inequality (2.77) contradicts (2.70). Therefore there exists an integer $k : 1 \leq k \leq N$ such that inequality (2.69) holds for $i = k$. We obtain the inequality

$$v(t) \leq v(\xi_k) < p(v(\xi_k)) \leq a(B), \quad t \in (t_k, t_{k+1}]$$

and

$$p(v(t_{k+1})) \leq v(t_{k+1}) < v(\xi_k) < a(B).$$

By induction we prove that

$$v(t) < a(B), \quad t > t_k,$$

and

$$p(v(t_i)) < a(B), \quad i = k+1, k+2\dots.$$

Therefore

$$v(t) < a(B) \quad \text{for } t > t_k. \tag{2.78}$$

From condition 2 and inequality (2.78) follows that

$$\|x(t)\| < B \quad \text{for} \ \ t \geq t_k. \tag{2.79}$$

We choose $T = N\rho_1$. The number T depends not only on α, but on t_0. Then $t_0 + T = (t_0 + \rho_1) + (N-1)\rho_1 \geq t_1 + (N-1)\rho_1 = t_1 + \rho_1 + (N-2)\rho_1 \geq t_2 + (N-2)\rho_1 \geq t_N \geq t_k$. Therefore inequality (2.79) holds for $t > t_0 + T$. □

We will apply some of the obtained sufficient conditions to investigate the boundedness of the solution of a linear impulsive hybrid differential equation.

Example 2.4.1. Consider the linear impulsive hybrid equation

$$x'(t) = ax(t) + b\lambda_k(x(k)) \quad \text{for} \quad t \in (k, k+1),\ k = 0, 1, 2, \dots, \tag{2.80}$$

$$x(k+0) = cx(k-0), \tag{2.81}$$

with initial condition

$$x(0) = x_0, \tag{2.82}$$

where $x \in \mathbf{R}$, a, b, c are constants.

Let the following conditions be fulfilled:

A1. Functions $\lambda_k \in C(\mathbf{R}, \mathbf{R})$ and there exists a constant $K > 0$ such that $\lambda_k(u) \leq Ku,\ k = 0, 1, 2, \dots$.

A2. Constants a, b, c are such that $b \geq 0$, $a + bK > 0$, $-1 < c < 1$ and $2\ln|c| + (a + bK) < 0$.

Define the functions $p(u) = \frac{u}{c^2}$, $F(t) = t$, $\quad G(t) \equiv a + bK$, $V(t, x) = \frac{x^2}{2}$, and $I_k(x) = (c-1)x$.

Function $V(t, x)$ satisfies the condition 2 of Theorem 2.4.2.

Let function $x(t) \in PC([0, \infty), \mathbf{R})$ be such that $|x(t)| > H$ and $|x(t)| > |x(k)|$ for $t \in (k, k+1)$. Then the inequality $p(V(t, x(t)) > V(k, x(k))$ holds for $t \in (k, k+1)$. We consider the derivative along the trajectory of the solution of the impulsive hybrid equation (2.80), (2.81). Then we obtain for $t \in (k, k+1)$

$$\begin{aligned}
& D^-_{(2.80),(2.81)} V(t, x(t), \lambda_k(x(k))) \\
& = \limsup_{\varepsilon \to 0-} \frac{1}{2\varepsilon} \{ \Big(x(t+\varepsilon; t, x(t), \lambda_k(x(k))) \Big)^2 - (x(t))^2 \} \\
& = \{ax(t; t, x(t), \lambda_k(x(k)) + b\lambda_k(x(k))\} x(t; t, x(t), \lambda_k(x(k)) \\
& \leq a(x(t))^2 + bKx(k)x(t) \leq (a + bK)\big(x(t)\big)^2 \\
= \quad & G(t)F(V(t, x(t))).
\end{aligned}$$

Therefore condition 3 of Theorem 2.4.2 is fulfilled.

From the choice of function $V(t,x)$ and function $p(t)$ follows that

$$p(V(k,x+I_k(x))) = \frac{(x+I_k(x))^2}{2c^2} = \frac{(cx)^2}{2c^2} = \frac{x^2}{2} = V(k,x)$$

Therefore condition 4 of Theorem 2.4.2 is fulfilled.

From the choice of function $F(t)$ and function $G(t)$ we obtain

$$\int_{k-1}^{k} G(s)\,ds = 2(a+bK)$$

and

$$\int_{\mu}^{p(\mu)} \frac{ds}{F(s)} = \int_{\mu}^{\frac{\mu}{c^2}} \frac{ds}{s} = -2ln|c|.$$

Therefore, condition 5 of Theorem 2.4.2 is satisfied for $A = -2ln|c| - (a+bK) > 0$.

According to Theorem 2.4.2 the solutions of the linear impulsive hybrid equation (2.80), (2.81) are uniform-ultimately bounded. □

2.5. Lyapunov Functions and Periodic Solutions of Impulsive Differential Equations

Lyapunov functions are extremely useful for qualitative investigations of different properties of the solutions of various types of differential equations. One of the unusual applications of the Lyapunov functions is the study of the existence and the behaviour of periodic solutions. The second method of Lyapunov with continuous functions is successfully applied in [56] for investigation the periodic solutions of differential-difference equations. Piecewise continuous analogues of classical Lyapunov functions give us the opportunity for a new approach in the qualitative study of the solutions of impulsive equations.

Let points $\{t_k\}_{k=-\infty}^{\infty}$ be fixed such that

$$t_{k+1} > t_k, \ \lim_{k\to-\infty} t_k = -\infty, \ \lim_{k\to+\infty} t_k = +\infty, \ t_{k+p} = t_k + T,$$

where p is a fixed number and $T > 0$ is a constant.

Consider the system of impulsive differential equations

$$x' = f(t,x(t)), \qquad t \neq t_k, \tag{2.83}$$

$$x(t_k+0) - x(t_k-0) = I_k(x(t_k-0)), \tag{2.84}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $(k = 0, \pm1, \pm2, \ldots)$.

Consider the scalar impulsive differential equation

$$u' = g(t,u), \qquad t \neq t_k, \tag{2.85}$$

$$u(t_k+0) = G_k(u(t_k-0)), \tag{2.86}$$

where $u \in \mathbf{R}$, and the functions $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $G_k : \mathbf{R} \to \mathbf{R}$, $(k = 0, \pm1, \pm2, \ldots)$ will be defined later.

We will say that conditions (H) are satisfied if:

H1. Function $f(t,x) \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ is a T-periodic in its first argument and it is Lipshitz in its second argument with a constant $M > 0$.

H2. Functions $I_k \in C(\mathbf{R}^n, \mathbf{R}^n)$, $(k = 0, \pm 1, \pm 2, \dots)$ are Lipshitz with constants $L_k > 0$ and $I_{k+p}(x) = I_k(x)$ for $x \in \mathbf{R}^n$.

We will obtain sufficient conditions for the existence of T-periodic solutions of the system of impulsive differential equations (2.83), (2.84).

Theorem 2.5.1. *Let the following conditions be fulfilled:*

1. *Conditions* (H) *are satisfied.*

2. *Function* $g(t,u) \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ *is Lipshitz in u, i.e. there exists a constant M such that* $|g(t,x) - g(t,y)| \le M|x - y|$ *for* $x, y \in \mathbf{R}$.

3. *Functions* $G_k \in C(\mathbf{R}, [0,\infty))$ *are Lipshitz, i.e. there exist constants* L_k *such that* $|G_k(x) - G_k(y)| \le L_k|x - y|$ *for* $x, y \in \mathbf{R}$ *and* $k = 0, \pm 1, \pm 2, \dots$.

4. *There exists a number* τ_0 *such that the scalar impulsive differential equation* (2.85), (2.86) *has a solution* $u(t)$ *for* $t \in [\tau_0, \tau_0 + T]$, *such that* $u(\tau_0) \ge u(\tau_0 + T)$ *and* $u(t) \ge H$ *for* $t \in [\tau_0, \tau_0 + T]$.

5. *There exists a function* $V(t,x)$ *from set* W *with the following properties*

(i) $a(||x||) \le V(t,x)$ *for* $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, *where* $a \in \Delta$;

(ii) *for* $(t,x) \in [\tau_0, \tau_0 + T] \times \mathbf{R}^n$, $t \ne t_k$ *and* $V(t,x) \ge H$ *the inequality*

$$D^-_{(2.83),(2.84)} V(t,x) \le g(t, V(t,x))$$

holds, where $H = const \ge \sup\{V(t,0) : \ t \in [\tau_0, \tau_0 + T]\}$;

(iii) $V(t_k + 0, x + I_k(x)) \le G_k(V(t_k, x))$ *for* $x \in \mathbf{R}^n$, $t_k \in [\tau_0, \tau_0 + T]$ *and* $V(t_k + 0, x + I_k(x)) \ge H$;

(iv) *function* $V(t,x)$ *is periodic in t with a period* T.

6. *Set* $\Sigma = \{x \in \mathbf{R}^n : \ V(\tau_0, x) \le u(\tau_0)\}$ *is convex.*

Then the system of impulsive differential equations (2.83), (2.84) *has a* T*-periodic solution.*

Proof. Consider the following two possible cases:

Case 1. Let for all natural number k inequality $\tau_0 \ne t_k$ hold. We will assume that $t_0 < \tau_0 < t_1$. Then the inequality $t_p < \tau_0 + T < t_{p+1}$ holds, i.e. $t_k \in [\tau_0, \tau_0 + T], k = 1, 2, \dots, p$.

We choose a point $x_0 \in \Sigma$ and denote the solution of the system of impulsive differential equations (2.83), (2.84) with initial condition $x(\tau_0) = x_0$ by $x(t; \tau_0, x_0)$, and the maximal existence interval by $J(\tau_0, x_0) \subset [\tau_0, \infty)$.

Denote $v(t) = V(t; x(t; \tau_0, x_0))$ for $t \in J(\tau_0, x_0)$.

We will prove that the inequality

$$v(t) \le u(t) \ \text{ for } \ t \in J(\tau_0, x_0) \cap [\tau_0, \tau_0 + T] \tag{2.87}$$

holds.

Assume the contrary and let

$$\eta = \inf\{t \in J(\tau_0, x_0) \cap [\tau_0, \tau_0 + T] : \ v(t) > u(t)\}. \tag{2.88}$$

Consider the following three cases:

Case 1.1. Let $\eta \neq t_k,\ k = 1,2,\dots,p$. From the properties of function $V(t,x)$ follows that function $v(t)$ has the following three properties:

P1. $v(\eta) = u(\eta)$;

P2. $v(t) \leq u(t)$ for $t \in [\tau_0, \eta)$;

P3. there exits a convergent sequence of points $\{\eta_m\}_1^\infty :\ \ \eta_m > \eta$, $\lim_{m\to\infty} \eta_m = \eta$, and the inequality $v(\eta_m) > u(\eta_m)$ holds.

From condition P3 follows that inequality

$$v'(\eta) \geq u'(\eta) \tag{2.89}$$

holds.

From condition 4 of Theorem 2.5.1 follows that

$$v(\eta) = u(\eta) \geq H. \tag{2.90}$$

Therefore inequality

$$v'(\eta) < g((\eta, v(\eta)) \tag{2.91}$$

holds.

From inequalities (2.89), (2.90), and (2.91) we obtain inequality

$$u'(\eta) \leq v'(\eta) < g(\eta, v(\eta) = g(\eta, u(\eta)). \tag{2.92}$$

Inequality (2.92) contradicts the fact that function $u(t)$ is a solution of the scalar impulsive differential equation (2.85), (2.86).

Case 1.2. Let $\eta = \tau_0$. From the definition of number η and the choice of point $x_0 \in \Sigma$ follows the validity of the equality $v(\eta) = u(\eta)$. Then there exists a point $\xi \in J(\tau_0, x_0) \cap [\tau_0, \tau_0 + T]$, $\xi > \tau_0$ such that $v(t) > u(t)$ for $t \in [\tau_0, \xi)$. Therefore there exists a sequence of points $\{\xi_m\}_1^\infty$ such that $\xi_m > \eta$, $\lim_{m\to\infty} \xi_m = \eta$ and $v(\xi_m) > u(\xi_m)$. Therefore the inequality (2.89) is true. As in the proof of case 1.1 we obtain a contradiction.

Case 1.3. Let there exist a natural number $k :\ 1 \leq k \leq p$ such that $\eta = t_k$. Then the inequalities

$$v(t_k + 0) > u(t_k + 0)$$

and

$$v(t) \leq u(t),\ \ t \in [\tau_0, t_k] \tag{2.93}$$

hold.

From condition 4 of Theorem 2.5.1 follows the validity of the inequality

$$v(t_k + 0) > u(t_k + 0) \geq H. \tag{2.94}$$

Consider the following two possible cases:

Case 1.3.1. Let $v(t_k) = u(t_k)$.

From condition (*iii*) and inequalities (2.93) and (2.94) we obtain

$$G_k(u(t_k)) = u(t_k + 0) < v(t_k + 0) \leq v(t_k) = G_k(u(t_k)). \tag{2.95}$$

Inequality (2.95) contradicts condition 3 of Theorem 2.5.1.

Case 1.3.2. Let $v(t_k) < u(t_k)$. The from condition (iii) inequality

$$G_k(u(t_k)) = u(t_k+0) < v(t_k+0) \leq G_k(v(t_k)) \leq G_k(u(t_k)) \tag{2.96}$$

holds.

Therefore inequality (2.87) holds.

We will prove that for $x_0 \in \Sigma$ the inclusion $J(\tau_0, x_0) \supset [\tau_0, \tau_0+T]$ is true.

Assume the contrary. Since the impulsive functions $I_k(x)$ are defined for all $x \in \mathbf{R}^n$ and according to the assumption there exists a point $\sigma \in [\tau_0, \tau_0+T]$ such that $\lim_{s\to\sigma-0} ||x(s;\tau_0,x_0)|| = \infty$.

From condition (i) and inequality (2.87) follows that $\lim_{s\to\sigma-0} u(s) = \infty$. The last equality contradicts the condition that function $u(t)$ is defined on the interval $[\tau_0, \tau_0+T]$.

Therefore, the solution $x(t;\tau_0,x_0)$ is defined on the interval $[\tau_0, \tau_0+T]$.

From inequality (2.87) and the periodicity of function $V(t,x)$ follows that

$$V(\tau_0, x(\tau_0+T;\tau_0,x_0)) = V(\tau_0+T, x(\tau_0+T;\tau_0,x_0)) = v(\tau_0+T)$$
$$\leq u(\tau_0+T) \leq u(\tau_0). \tag{2.97}$$

Inequality (2.97) implies that the operator $Q : x_0 \to x(\tau_0+T;\tau_0,x_0)$ transforms the set Σ into itself. From conditions 4, 5, and 6 of Theorem 2.5.1 it follows that set Σ is nonempty, bounded, closed, and convex. According to the Brauer's Fixed Point Theorem the operator $Q : \Sigma \to \Sigma$ has a fixed point on the set Σ , i.e. there exists a point $x_0 \in \Sigma$ such that $x_0 = x(\tau_0+T;t_0,x_0)$.

Case 2. Let there exist a natural number k such that $\tau_0 = t_k$. We assume that $\tau_0 = t_0$. As in the proof of case 1, if we assume that $v(t_0) < u(t_0)$ then from the conditions of the functions $v(t)$ and $G_0(u)$ follows that

$$v(t_0+0) \leq G_0(v(t_0)) < G_0(u(t_0)) = u(t_0+0).$$

The last inequality implies that $\eta > t_0$. □

Theorem 2.5.2. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, 3, *and* 6 *of Theorem* 2.5.1 *are satisfied.*

2. *There exists a number* τ_0 *such that the scalar impulsive differential equation* (2.85), (2.86) *has a solution* $u(t)$ *for* $t \in [\tau_0, \tau_0+T]$, *such that the inequalities* $u(\tau_0) \geq u(\tau_0+T)$ *and* $u(t) \geq b(H)$ *hold for* $t \in [\tau_0, \tau_0+T]$.

3. *There exists a function* $V(t,x)$ *from set* W *such that*

(i) $a(||x||) \leq V(t,x) \leq b(||x||)$ *for* $(t,x) \in \mathbf{R}\times\mathbf{R}^n$, *where* $a \in \Delta$, $b \in CIP$;

(ii) *for* $(t,x) \in [\tau_0, \tau_0+T]\times\mathbf{R}^n$, $t \neq t_k$, $||x|| \geq H$ *the inequality*

$$D^-_{(2.83),(2.84)}V(t,x) \leq g(t, V(t,x))$$

holds, where $H = const \geq \sup\{V(t,0) :\ t \in [\tau_0, \tau_0+T]\}$;

(iii) $V(t_k+0, x+I_k(x)) \leq G_k(V(t_k,x))$ *for* $x \in \mathbf{R}^n$, $t_k \in [\tau_0, \tau_0+T]$ *and* $||x+I_k(x)|| \geq H$;

(iv) *function* $V(t,x)$ *is periodic in* t *with a period* T.

Then the system of impulsive differential equations (2.83), (2.84) *has a T-periodic solution.*

Proof. The proof is analogous to the proof of Theorem 2.5.1, where we use the inequality $||x(\eta;t_0,x_0)|| \geq H$ instead of inequality (2.90). □

We will obtain sufficient conditions for the existence of periodic solutions of the system of impulsive differential equations (2.83), (2.84).

Theorem 2.5.3. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, 3, *and* 6 *of Theorem* 2.5.1 *are satisfied.*

3. *There exists a number* τ_0 *and a function* $L \in C(\mathbf{R},[0,\infty))$ *such that scalar impulsive differential equation* (2.85), (2.86) *has a solution* $u(t)$ *for* $t \in [\tau_0,\tau_0+T]$ *such that the inequalities* $u(\tau_0) \geq u(\tau_0+T)$ *and* $u(t) \geq L(u(t))$ *hold for* $t \in [\tau_0,\tau_0+T]$.

4. *There exists a function* $V(t,x)$ *from set* W *such that*

(*i*) $V(\tau_0,x) \leq u(\tau_0)$;

(*ii*) $a(||x||) \leq V(t,x)$ *for* $(t,x) \in \mathbf{R}\times\mathbf{R}^n$, *where* $a \in \Delta$;

(*iii*) *for* $(t,x) \in [\tau_0,\tau_0+T]\times\mathbf{R}^n$, $t \neq t_k$ *and* $V(t,x) \leq L(V(t,x))$ *the inequality*

$$D^-_{(2.83),(2.84)}V(t,x) \leq g(t,V(t,x))$$

holds, where $H = const \geq \sup\{V(t,0): \ t \in [\tau_0,\tau_0+T]\}$;

(*iv*) $V(t_k+0,x+I_k(x)) \leq G-k(V(t_k,x))$ *for* $x \in \mathbf{R}^n$, $t_k \in [\tau_0,\tau_0+T]$;

(*v*) *function* $V(t,x)$ *is periodic in* t *with a period of* T.

Then the system of impulsive differential equations (2.83), (2.84) *has a T-periodic solution.*

Proof. The proof of Theorem 2.5.3 is analogous to the proof of Theorem 2.5.1. In this case point η in equality (2.88) is defined by

$$\eta = \sup\{t \in J(\tau_0,x_0)\cap[\tau_0,\tau_0+T]: \ v(s) \leq u(s), \ s \in [\tau_0,t]\}. \qquad \square$$

Chapter 3

Monotone-Iterative Techniques for Impulsive Equations

Since the set of nonlinear problems that solutions could be presented as well known functions is too narrow, one needs to exploit various approximate methods. There are different analytic approximate methods applied to various types of differential equations ([2], [19], [31], [32], [79], [110]).

In this chapter the monotone-iterative techniques are applied to different types of impulsive equations. The monotone-iterative techniques combine the method of lower and upper solutions with an appropriate monotone method. The class of the impulsive equations to which these techniques could be applied is comparatively wide because the algorithm for constructing successive approximations is very simple and the conditions for the right part of the equations are natural. This technique is applied successfully to different types of differential equations without impulses ([36], [92], [94], [102], [116], the monograph [88], and the cited therein references).

We note that similar results to those proved in this section are published in [9], [12], [13], [68], [69], [70], [71], [72].

3.1. Monotone-Iterative Techniques for The initial Value Problem for Impulsive Differential-Difference Equations with Variable Moments of Impulses

We will begin considerations with the initial value problem. To extend the set of impulsive equations for which the method is applicable, we will consider impulsive differential-difference equations. We will apply this method to the initial value problem for nonlinear scalar impulsive differential-difference equations in the case of variable moments of impulses, i.e. the impulses occur when the integral curve of the solution intersects one of the initially given hypersurfaces.

In this section we will consider only one hypersurface. This way we will avoid some complicated notations and technical difficulties and will keep all ideas in the proofs.

Let set $\sigma = \{(t,x) \in \mathbf{R}\times\mathbf{R} : \ t = \tau(x)\}$ be given, where function $\tau(x) \in C(\mathbf{R},\mathbf{R})$.

Consider the initial value problem for the scalar impulsive nonlinear differential-difference equation

$$x' = f(t, x(t), x(t-h)) \quad \text{for } t > t_0,\ t \neq \tau(x(t)), \tag{3.1}$$

$$x(t+0) - x(t-0) = I(x(t)) \quad \text{for } t = \tau(x(t)), \tag{3.2}$$

$$x(t) = \varphi(t) \quad \text{for } t \in [t_0 - h, t_0], \tag{3.3}$$

where $x \in \mathbf{R}$, $f : [t_0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $I : \mathbf{R} \to \mathbf{R}$, $\varphi : [t_0 - h, t_0] \to \mathbf{R}$, $h = const > 0$, $t_0 < T$, $T = const < \infty$.

Consider the corresponding initial value problem for the scalar nonlinear differential-difference equations without impulses

$$x' = f(t, x(t), x(t-h)) \quad \text{for } t > t_0, \tag{3.4}$$

$$x(t) = \varphi(t) \quad \text{for } t \in [t_0 - h, t_0]. \tag{3.5}$$

We will denote the solution of the initial value problem for the impulsive nonlinear differential-difference equation (3.1), (3.2), (3.3) by $x(t; t_0, \varphi)$, and the solution of the corresponding initial value problem for the nonlinear differential-difference equation without impulses (3.4), (3.5) by $X(t; t_0, \varphi)$.

We will apply the monotone iterative technique on the interval $[t_0 - h, T]$, $t_0 < T < \infty$.

We will say, that conditions (H) are satisfied if:

H1. Function $f \in C([t_0, T] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$.

H2. Function f is Lipshitz, i.e. there exist constants $L_1 > 0$ and $L_2 > 0$ such that for every two points $(x, y), (\xi, \eta) \in \mathbf{R} \times \mathbf{R}$ and for all $t \in [t_0, T]$ the inequality

$$|f(t, x, y) - f(t, \xi, \eta)| \leq L_1 |x - \xi| + L_2 |y - \eta|$$

holds.

H3. There exists a constant $M > 0$ such that $|f(t, x, y)| \leq M$ for $(t, x, y) \in [t_0, T] \times \mathbf{R} \times \mathbf{R}$.

H4. Function $\tau \in C(\mathbf{R}, [t_0, T])$ is Lipshitz with a constant $L > 0$, where $0 \leq L < \frac{1}{M}$.

H5. Inequality $\tau(x + I(x)) < \tau(x)$, $x \in \mathbf{R}$ holds.

H6. Function $\varphi \in C([t_0 - h, t_0], \mathbf{R})$.

One of the unique phenomena of the solutions of impulsive equations with variable moments of impulses is the “beating”, i.e. the case when the integral curve of the solution intersects the same set σ many times (finite or infinity many times). This phenomenon could be the reason for interruption of the existence of the solution. Therefore, it is necessary the obtaining of sufficient conditions for absence of “beating” of the solutions of the initial value problem (3.1), (3.2), (3.3), i.e. sufficient conditions for not more than one intersection point of the set σ and the integral curve of the solution on the interval $[t_0, T]$.

Lemma 3.1.1. *Let conditions* (H) *be satisfied.*

Then

(i) *The integral curve of the initial value problem for the impulsive equation* (3.1), (3.2), (3.3) *meets only once curve* σ *in* $t_0 < t < T$.

(ii) *The solution of the initial value problem for the impulsive equation* (3.1), (3.2), (3.3) *exists on* $[t_0 - h, T]$.

Proof. (*i*). From conditions H1, H2, and H6 follows that the initial value problem for the differential-difference equation (3.4), (3.5) has unique solution $X(t;t_0,\varphi)$ on $[t_0-h,T]$. Assume that for $t\in(t_0,T)$ the inequality $t\neq\tau(X(t;t_0,\varphi))$ holds. Then $x(t;t_0,\varphi)=X(t;t_0,\varphi)$ for $t_0-h\le t\le T$ and it is a continuous function. Therefore the function $\psi(t)=t-\tau(x(t;t_0,\varphi))$ is continuous on $[t_0,T]$ and according to the assumption the inequality $\psi(t)\neq 0$ holds on the interval $[t_0,T]$. At the same time the inequalities

$$\psi(t_0)=t_0-\tau(x(t_0;t_0,\varphi))<0,\ \ \psi(T)=T-\tau(x(T;t_0,\varphi))>0$$

hold.

Therefore, there exists a point $\tau_1\in(t_0,T)$ such that the equality $\psi(\tau_1)=0$ holds, i.e. $\tau_1=\tau(x(\tau_1;t_0,\varphi))$. The obtained contradiction implies that the integral curve of the solution of the initial value problem for impulsive equation (3.1), (3.2), (3.3) meets curve σ at moment τ_1.

Assume that the integral curve $(t,x(t;t_0,\varphi))$ intersects more than once the curve σ and let τ_1 and τ_2 : $\tau_1<\tau_2$ be two consecutive moments of intersection. From conditions H3, H4, and H5 we obtain the inequalities

$$\begin{aligned}
\tau_2-\tau_1 &= \tau(x(\tau_2;t_0,\varphi))-\tau(x(\tau_1;t_0,\varphi))\\
&< \tau(x(\tau_2;t_0,\varphi))-\tau(x(\tau_1;t_0,\varphi)+I(x(\tau_1;t_0,\varphi)))\\
&= \tau(x(\tau_2;t_0,\varphi))-\tau(x(\tau_1+0;t_0,\varphi))\\
&\le L|x(\tau_2;t_0,\varphi)-x(\tau_1;t_0,\varphi)|\\
&= L|\int_{\tau_1}^{\tau_2} f(s,x(s;t_0,\varphi),x(s-h;t_0,\varphi))ds|\\
&\le ML(\tau_2-\tau_1)<\tau_2-\tau_1.
\end{aligned}$$

The obtained contradiction proves the validity of (*i*).

Proposition (*ii*) follows from conditions H1, H2, H6, and (*i*). □

Definition 19. We will say that function $v(t)$ is a lower (upper) solution of the initial value problem for the impulsive equation (3.1), (3.2), (3.3) if

$$v'\le\ (\ge)\ f(t,v(t),v(t-h))\quad \text{for } t>t_0,\ t\neq\tau(v(t)), \tag{3.6}$$

$$v(t+0)-v(t-0)\le\ (\ge)\ I(v(t))\quad \text{for } t=\tau(v(t)), \tag{3.7}$$

$$v(t)=\varphi(t)\quad \text{for } t\in[t_0-h,t_0]. \tag{3.8}$$

Similarly, we can define lower and upper solution of the initial value problem for the differential-difference equation without impulses (3.4), (3.5).

We will obtain sufficient conditions for absence of *beating* of the lower (upper) solutions of the initial value problem for the impulsive equation (3.1), (3.2), (3.3).

Lemma 3.1.2. *Let the following conditions be fulfilled:*

1. *Conditions* H1–H4 *and* H6 *are satisfied.*
2. *Function* $\tau(x)$ *is increasing (decreasing).*

Then

(i) *the integral curve of the lower* $(upper)$ *solution of the initial value problem for the impulsive equation* (3.1), (3.2), (3.3) *intersects only once the curve* σ *for* $t_0 < t < T$.

(ii) *The lower (upper) solutions of the initial value problem for the impulsive equation* (3.1), (3.2), (3.3) *exists on the interval* $[t_0 - h, T]$.

Proof. (i). Let function $\tau(x)$ be increasing and function $v(t;t_0,\varphi)$ be a lower solution of the initial value problem for the impulsive equation (3.1), (3.2), (3.3), defined for $t \geq t_0$. If the integral curve $(t, v(t;t_0,\varphi))$ does not intersect curve σ on the interval (t_0,T), then the function $\psi(t) = t - \tau(v(t;t_0,\varphi))$ is continuous on the interval $[t_0,T]$ and $\psi(t_0) < 0$ and $\psi(T) > 0$. Therefore there exists a point $\tau_1 \in (t_0,T)$ such that $\psi(\tau_1) = 0$, i.e.

$$\tau_1 = \tau(v(\tau_1;t_0,\varphi)).$$

The last equality implies that curve $(t, v(t;t_0,\varphi))$ intersects σ for $t_0 < t < T$, i.e. the integral curve of the lower solution of the initial value problem for the impulsive equation (3.1), (3.2), (3.3) intersects at least once the curve σ.

Assume that the integral curve $(t, v(t;t_0,\varphi))$ has at least two common points with the curve σ and let τ_1 and τ_2 be two consecutive points of intersection and $\tau_1 < \tau_2$. From the fact that the function $\tau(x)$ is increasing and the inequality

$$0 < \tau_2 - \tau_1 = \tau(v(\tau_2;t_0,\varphi)) - \tau(v(\tau_1;t_0,\varphi))$$

it follows that $v(\tau_2;t_0,\varphi) > v(\tau_1;t_0,\varphi)$. From condition 2 it follows that $\tau(x) > \tau(x + I(x))$, i.e. $x > x + I(x)$. From the last inequality we obtain that $I(x) < 0$ for $x \in \mathbf{R}$. Therefore inequality

$$v(\tau_1;t_0,\varphi) > v(\tau_1 + 0;t_0,\varphi) = v(\tau_1;t_0,\varphi) + I(v(\tau_1;t_0,\varphi))$$

holds. Therefore,

$$\begin{aligned}
\tau_2 - \tau_1 &= \tau(v(\tau_2;t_0,\varphi)) - \tau(v(\tau_1;t_0,\varphi)) \\
&< \tau(v(\tau_2;t_0,\varphi)) - \tau(v(\tau_1;t_0,\varphi) + I(v(\tau_1;t_0,\varphi))) \\
&= \tau(v(\tau_2;t_0,\varphi)) - \tau(v(\tau_1 + 0;t_0,\varphi)) \\
&\leq L|v(\tau_2;t_0,\varphi) - v(\tau_1;t_0,\varphi)| \\
&\leq L|\int_{\tau_1}^{\tau_2} f(s, v(s;t_0,\varphi), v(s-h;t_0,\varphi))ds| \\
&\leq ML(\tau_2 - \tau_1) < \tau_2 - \tau_1.
\end{aligned}$$

The obtained contradiction proves proposition (i) of Lemma 3.1.2.

Proposition (ii) follows from conditions H1, H2, H6, and (i).

The case of upper solutions and a decreasing function $\tau(x)$ can be proved analogously. □

As a corollary of Lemma 3.1.2 we obtain the following result about lower (upper) solutions of the initial value problem for the differential-difference equation without impulses (3.4), (3.5):

Lemma 3.1.3. *Let the following conditions be fulfilled:*

1. *Conditions* H1–H4 *and* H6 *are satisfied.*

2. *Function* $\tau(x)$ *is monotone increasing* $($*decreasing*$)$.

Then the integral curve of the lower $($*upper*$)$ *solution of the initial value problem for the differential-difference equation without impulses* (3.4), (3.5) *has unique common point with the curve* σ *on the interval* (t_0, T).

Let functions $v, w \in PC^1([t_0, T])$ and $v(t) \le w(t)$ for $t \in [t_0, T]$. Denote

$$S([t_0,T],v,w) = \{u : [t_0,T] \to \mathbf{R}, v(t) \le w(t), t \in [t_0,T]\}.$$

Consider the scalar differential- difference equation

$$x' = -ax(t) - bx(t-h) \ \text{ for } t \in [t_0, T], \tag{3.9}$$

with the initial condition

$$x(t) = 0, \quad \text{for } t_0 - h \le t \le t_0, \tag{3.10}$$

where a, b are constants.

In further investigations we will use the following result:

Lemma 3.1.4. *Let the following conditions be fulfilled:*

1. *Constants a and b are positive.*

2. *Inequality* $(a+b)(T-t_0) \le 1$ *holds.*

Then all lower solutions of the initial value problem (3.9), (3.10) *are nonnegative.*

Proof. Let $x = x(t)$ be an arbitrary lower solution of the initial value problem (3.9), (3.10). Denote

$$t_1 = \inf\{t \in [t_0, T] : \ \ x(t) \ne 0\}.$$

It is clear that $x(t_1) = 0$.

Assume that there exists a point $t_2 \in (t_1, T)$ such that $x(t_2) > 0$.

Consider the following three cases:

Case 1. There exists a constant $h_1 \in (t_0, T - t_1)$ such that $x(t) \le 0$ for $t \in [t_1, t_1 + h_1]$. Since $x(t_2) > 0$, we conclude that there exists at least one point on the interval $[t_1 + h_1, t_2)$, that vanishes the solution $x(t)$. Let

$$t_3 = \inf\{t \in [t_1 + h_1, t_2) : \ \ x(t) = 0\}.$$

Then $x(t_3) = 0$. Denote $m = \min\{x(t) : \ t_1 \le t \le t_3\}$. Then $m < 0$. Let $m = x(t_4)$, where $t_1 < t_4 < t_3$. The inequalities

$$\begin{aligned} -m &= -x(t_4) = x(t_3) - x(t_4) = \int_{t_4}^{t_3} \frac{dx(t)}{dt} dt \\ &\le \int_{t_4}^{t_3} (-ax(t) - bx(t-h)) dt \\ &\le \int_{t_4}^{t_3} (-am - bm) dt = -m(a+b)(t_3 - t_4) \\ &< -m(a+b)(T - t_0) \end{aligned}$$

hold. Therefore we obtain the inequality $1 < (a+b)(T-t_0)$, which contradicts condition 2 of Lemma 3.1.4.

Case 2. There exists a constant $h_1,\ 0 < h_1 < \min\{h, T-t_1\}$ such that $x(t) \geq 0$ for $t_1 \leq t \leq t_1 + h_1$. On this interval the solution $x = x(t)$ is a decreasing function since

$$x'(t) \leq -ax(t) - bx(t-h) = -ax(t) \leq 0. \tag{3.11}$$

From the choice of point t_1 and the conditions of this case it follows that there exists a point $t \in [t_1, t_1+h_1]$ such that $x(t) > 0$. Therefore at that point the inequality (3.11) is strict. From this fact and the equality $x(t_1) = 0$ we conclude that $x(t) < 0$ that contradicts the assumptions of the case.

Case 3. There exists an increasing sequence $\{t_k^-\}_1^\infty$ such that $\lim_{n\to\infty} t_n^- = t_1$ and the inequality $x(t_n^-) \leq 0$ holds for $n = 1, 2, \dots$. At the same time there exists a sequence $\{t_k^+\}_1^\infty$ such that $t_1^- < t_n^+ < t_{k+1}^-$ and the inequality $x(t_n^+) > 0$ holds for $n = 1, 2, \dots$. Therefore in this case there exists a sequence $\{t_k^0\}_1^\infty$, such that the inequalities $t_n^- < t_n^0 < t_{n+1}^-$, $x(t_n^0) > 0$ and $\frac{dx(t_n^0)}{dt} = 0$ hold for $n = 1, 2, \dots$. Since $\lim_{n\to\infty} t_n^0 = t_1$, then we conclude that there exists a number n_0, such that the inequality $t_1 < t_n^0 < t_1 + h$ holds for $n > n_0$. From inequality for the lower solution, that corresponds to the equation (3.9) for $n > n_0$ we obtain the following contradiction

$$0 = \frac{dx(t_n^0)}{dt} \leq -ax(t_n^0) - bx(t_n^0 - h) = -ax(t_n^0) < 0.$$

Therefore, $x(t) \leq 0$ for $t \in (t_1, T)$. □

We will give an algorithm for constructing successive approximation of the solution of the initial value problem for the impulsive differential-difference equation (3.1), (3.2), (3.3). For this purpose we will use the lower solutions of the corresponding initial value problem for the differential-difference equation without impulses (3.4), (3.5) and appropriate monotone method.

We will say that the conditions (A) are satisfied if:

A1. Function $v_0 \in C([t_0-h, T], \mathbf{R})$ is a lower solution the initial value problem for the differential-difference equation without impulses (3.4), (3.5).

A2. Function $w_0 \in C([t_0-h, T], \mathbf{R})$ is an upper solution of the initial value problem for the differential-difference equation without impulses (3.4), (3.5).

A3. Inequality $(L_1 + L_2)(T - t_0) \leq 1$ holds.

A4. Function $I \in C(\mathbf{R}, \mathbf{R})$.

The main result in this section is the following theorem:

Theorem 3.1.1. *Let the following conditions be fulfilled:*

1. *Conditions* H1–H6, A1, A3 *and* A4 *are satisfied.*

2. *Function* $\tau(x)$ *is increasing.*

There exists a sequence of functions $\{u_n(t)\}_0^\infty$ *such that* $u_n(t) : [t_0-h, T] \to \mathbf{R}$ *and:*

a/ Functions $u_n(t)$ *are lower solutions of the initial value problem for the impulsive differential-difference equation* (3.1), (3.2), (3.3)*;*

b/ The sequence of functions $\{u_n(t)\}_0^\infty$ *is nondecreasing, i.e.*

$$u_1(t) \leq u_2(t) \leq u_3(t) \leq \dots \quad \text{for } t_0 - h \leq t \leq T;$$

c/ Limit $u(t)=\lim_{n\to\infty} u_n(t)$ *for* $t_0-h\le t\le T$ *exists;*

d/ Function $u(t)$ *is a solution to the initial value problem for the impulsive differential-difference equation* (3.1), (3.2), (3.3) *on the interval* $[t_0-h,T]$.

Proof. Let point $t_1\in[t_0,T]$, function $\varphi_1\in C([t_1-h,t_1],\mathbf{R})$, and function $x_1\in S([t_1-h,T],v_0,w_0)$.

Consider the initial value problem for the linear differential-difference equation

$$x'=-L_1x(t)-L_2x(t-h)+F(t,x_1),\quad t\in[t_1,T] \tag{3.12}$$

with initial condition

$$x(t)=\varphi_1(t),\quad t_1-h\le t\le t_1, \tag{3.13}$$

where

$$F(t,x_1)=f(t,x_1(t),x_1(t-h))+L_1x_1(t)+L_2x_1(t-h).$$

The initial value problem for linear differential-difference equation (3.12), (3.13) has an unique solution for any fixed function $x_1(t)\in S([t_1-h,T],v_0,w_0)$. Denote this solution by $x_2(t)$.

Define operator Ω with the help of equation

$$x_2=\Omega(t_1,\varphi_1,x_1),$$

i.e. to any triple (t_1,φ_1,x_1) such that $t_1\in[t_0,T]$, $\varphi_1\in C([t_1-h,t_1],\mathbf{R})$ and $x_1\in S([t_1-h,T],v_0,w_0)$ we correspond the unique solution of the initial value problem (3.12), (3.13).

Consider the following initial value problem for the nonlinear differential-difference equation

$$x'=f(t,x(t),x(t-h))\quad\text{for}\ \ t_1\le t\le T, \tag{3.14}$$

$$x(t)=\varphi_1(t)\quad\text{for}\ \ t\in[t_1-h,t_1]. \tag{3.15}$$

Let $x_1(t)$ be a lower solution of problem (3.14), (3.15). Then the operator Ω has the following properties:

P1. If $x_2=\Omega(t_1,\varphi_1,x_1)$, then $x_2(t)\ge x_1(t)$ for $t_1-h\le t\le T$;

P2. Function x_2 is a lower solution of the problem (3.14), (3.15);

P3. If $x_1^{(1)},x_1^{(2)}\in S([t_1-h,T],v_0,w_0)$, $x_1^{(1)}(t)\le x_1^{(2)}(t)$ for $t_1-h\le t\le t_1$ and $x_2^{(1)}=\Omega(t_1,\varphi_1,x_1^{(1)})$, $x_2^{(2)}=\Omega(t_1,\varphi_1,x_1^{(2)})$, then $x_2^{(1)}(t)\le x_2^{(2)}(t)$ for $t_1-h\le t\le T$.

Indeed, to prove property P1 we set $x(t)=x_1(t)-x_2(t)$ for $t_1-h\le t\le T$. Then we obtain

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{dx_1(t)}{dt}-\frac{dx_2(t)}{dt}\\ &\le f(t,x_1(t),x_1(t-h))+L_1x_2(t)+L_2x_2(t-h)-F(t,x_1)\\ &= -L_1\Big(x_1(t)-x_2(t)\Big)-L_2\Big(x_1(t-h)-x_2(t-h)\Big)\\ &= -L_1x(t)-L_2x(t-h),\ \ t_1<t\le T.\end{aligned}$$

The following equality

$$x(t)=0,\ \ t_1-h\le t\le t_1$$

holds. Then according to Lemma 3.1.4 the inequality $x(t) \leq 0$ holds for $t_1 - h \leq t \leq T$ that proves the validity of property P1.

The validity of properties P2 and P3 can be proved analogously.

According to Lemma 3.1.3 there exists an unique point $\tau_{v_0} \in (t_0, T)$ such that

$$\tau_{v_0} = \tau(v_0(\tau_{v_0})),$$

i.e. the integral curve $(t, v_0(t))$ intersects the curve σ only at the moment τ_{v_0} on the interval (t_0, T).

Let $v_0^+(t)$ be a lower solution of the initial value problem for the differential-difference equation without impulses (3.14), (3.15), where $t_1 = \tau_{v_0}$, and the function φ_1 is defined by

$$\varphi_1(t) = \begin{cases} v_0(t) & \text{for } \tau_{v_0} - h \leq t < \tau_{v_0} \\ v_0(t) + I(v_0(t)) & \text{for } t = \tau_{v_0}. \end{cases}$$

The equality $v_0^+(t) = v_0(t)$ holds for $\tau_{v_0} - h \leq t < \tau_{v_0}$. As in the proof of Lemma 3.1.2, from condition H5 and monotonicity of the function τ follows that $I(x) < 0$ for $x \in \mathbf{R}$. Therefore

$$v_0^+(\tau_{v_0}) = \varphi_1(\tau_{v_0}) = v_0(\tau_{v_0}) + I(v_0(\tau_{v_0})) < v_0(\tau_{v_0}).$$

Therefore the lower solution $v_0^+(t)$ of the initial value problem for the differential-difference equation (3.14), (3.15) is chosen such that

$$v_0^+(t) \leq v_0(t), \quad \tau_{v_0} - h \leq t < T.$$

Set

$$u_0(t) = \begin{cases} v_0(t) & \text{for } t_0 - h \leq t < \tau_{v_0} \\ v_0^+(t) & \text{for } \tau_{v_0} \leq t < T. \end{cases}$$

Function $u_0(t)$ is a lower solution of the impulsive differential-difference equation (3.1), (3.2), (3.3).

Since $v_0(t)$ is a lower solution of the initial value problem for the differential-difference equation without impulses (3.14), (3.15), it is possible to define a function $v_1 = \Omega(t_0, \varphi, v_0)$, i.e. v_1 is a solution of the problem (3.12), (3.13) for $t_1 = t_0$, $\varphi_1(t) = \varphi(t)$ and $x_1(t) = v_0(t)$. From properties P1, P2, and P3 of the operator Ω it follows that $v_1(t)$ is a lower solution of the problem (3.14), (3.15) for $t_1 = t_0$ and $\varphi_1(t) = \varphi(t)$. Therefore, the inequality

$$v_1(t) \geq v_0(t), \quad t_0 - h \leq t \leq T \tag{3.16}$$

holds.

According to Lemma 3.1.3 there exists an unique point $\tau_{v_1} \in (t_0, T)$ such that

$$\tau_{v_1} = \tau(v_0(\tau_{v_1})),$$

i.e. the integral curve $(t, v_1(t))$ intersects the curve σ only at point τ_{v_1} on the interval (t_0, T).

We will prove that $\tau_{v_1} > \tau_{v_0}$. If we assume the contrary, i.e. the inequality $\tau_{v_1} \leq \tau_{v_0}$ holds, then from inequality (3.16) and the monotonicity of function τ we obtain

$$\psi(\tau_{v_1}) = \tau_{v_1} - \tau(v_0(\tau_{v_1})) = \tau(v_1(\tau_{v_1})) - \tau(v_0(\tau_{v_1})) \geq 0.$$

In addition the inequality

$$\psi(t_0) = t_0 - \tau(v_0(t_0)) < 0$$

holds.

Therefore, there exists a point $\tau^* \in (t_0, \tau_{v_1}]$ such that $\psi(\tau^*) = 0$, i.e.

$$\tau^* = \tau(v_0(\tau^*)).$$

From the last equality we conclude that curves $(t, v_0(t))$ and σ have a common point $\tau^*: \ \tau^* \le \tau_{v_1} < \tau_{v_0}$, i.e. the integral curve $(t, v_0(t))$ intersects curve σ at two different points τ^* and τ_{v_0}, that contradicts the conclussion of Lemma 3.1.3.

We consider function $v_1^+ = \Omega(\tau_{v_1}, \varphi_1, v_0^+)$, where

$$\varphi_1(t) = \begin{cases} v_1(t) & \text{for } \tau_{v_1} - h \le t < \tau_{v_1} \\ v_1(t) + I(v_1(t)) & \text{for } t = \tau_{v_1}. \end{cases} \tag{3.17}$$

According to property P2 of operator Ω function v_1^+ is a lower solution of the problem (3.14), (3.15), where $t_1 = \tau_{v_1}$, and the function φ_1 is defined by the equality (3.17). From property P1 of operator Ω follows that for $\tau_{v_1} \le t \le T$ the inequality $v_1^+(t) \ge v_0^+(t)$ holds.

Define the function

$$u_1(t) = \begin{cases} v_1(t) & \text{for } t_0 - h \le t < \tau_{v_1} \\ v_1^+(t) & \text{for } \tau_{v_1} \le t < T. \end{cases}$$

Function $u_1(t)$ is a lower solution of the initial value problem for the impulsive differential-difference equation (3.1), (3.2), (3.3) and the inequality

$$u_1(t) \ge u_0(t), \quad t_0 - h \le t \le T$$

holds.

Assume that for a natural number n the functions $v_{n-1} : \ [t_0 - h, T] \to \mathbf{R}$, $v_{n-1}^+ : [t_0 - h, T] \to \mathbf{R}$, and the functions

$$u_{n-1}(t) = \begin{cases} v_{n-1}(t) & \text{for } t_0 - h \le t < \tau_{v_{n-1}} \\ v_{n-1}^+(t) & \text{for } \tau_{v_{n-1}} \le t < T \end{cases}$$

are constructed.

Set $v_n = \Omega(t_0, \varphi, v_{n-1})$. Let $\tau_{v_n} = \tau(v_n(\tau_{v_n}))$. According to Lemma 3.1.3 there is only one point that vanishes the function $\psi(t) = t - \tau(v_n(t))$. It is easy to prove the validity of the following inequality

$$\tau_{v_n} > \tau_{v_{n-1}}. \tag{3.18}$$

Define function $v_n^+ = \Omega(\tau_{v_n}, \varphi_1, v_{n-1}^+)$, where

$$\varphi_1(t) = \begin{cases} v_n(t) & \text{for } \tau_{v_n} - h \le t < \tau_{v_n} \\ v_n(t) + I(v_n(t)) & \text{for } t = \tau_{v_n}. \end{cases} \tag{3.19}$$

At the end we set

$$u_n(t) = \begin{cases} v_n(t) & \text{for } t_0 - h \le t < \tau_{v_n} \\ v_n^+(t) & \text{for } \tau_{v_n} \le t \le T. \end{cases}$$

Function $u_n(t)$ is a lower solution of the initial value problem of the impulsive differential-difference equation (3.1), (3.2), (3.3) and the inequality

$$u_n(t) \geq u_{n-1}(t), \quad t_0 - h \leq t \leq T \tag{3.20}$$

holds.

We construct a sequence $\{\tau_{v_j}\}_{j=0}^{\infty}$ and a sequence $\{u_j(t)\}_{j=0}^{\infty}$ of lower solutions of the initial value problem for the impulsive differential-difference equation (3.1), (3.2), (3.3) such that the inequalities (3.18) and (3.20) hold.

Since the sequence $\{\tau_{v_j}\}_{j=0}^{\infty}$ is increasing and it is bounded from above by a constant T, the sequence is convergent and let its limit be $\tau_v \in (t_0, T]$.

Consider the sequence $\{v_j(t)\}_{j=0}^{\infty}$ of continuous functions on the interval $[t_0 - h, T]$. According to the definition of functions $v_j(t)$ and property P1 of operator Ω we conclude that this sequence is increasing, i.e.

$$v_n(t) \geq v_{n-1}(t), \quad t_0 - h \leq t \leq T, \ \ n = 1, 2, \ldots.$$

We will prove that the sequence of functions $\{v_j(t)\}_{j=0}^{\infty}$ is bounded from above. Indeed, for $t_0 - h \leq t \leq t_0$ the equalities $v_n(t) = \varphi(t), \ \ n = 1, 2, \ldots$ hold. Therefore,

$$v_n(t) \leq M_\varphi = \max\{\varphi(t) : \ \ t \in [t_0 - h, t_0]\}, \ \ n = 1, 2, \ldots.$$

For $t \in (t_0, T]$ the inequality

$$\begin{aligned} v_n(t) \qquad & = v_n(t_0) + \int_{t_0}^{t} \Big(L_1(v_{n-1}(s) - v_n(s)) \\ + \quad & L_2(v_{n-1}(s-h) - v_n(s-h)) \Big) ds \\ + \quad & \int_{t_0}^{t} f(s, v_{n-1}(s), v_{n-1}(s-h)) ds \leq \varphi(t_0) \\ + \quad & \int_{t_0}^{t} f(s, v_{n-1}(s), v_{n-1}(s-h)) ds \\ \leq \quad & \varphi(t_0) + M(t - t_0) \leq M_\varphi + M(T - t_0) \end{aligned} \tag{3.21}$$

holds.

Following the above we obtain that

$$v_n(t) \leq M_\varphi + M(T - t_0), \quad t_0 - h \leq t \leq T.$$

Therefore, the sequence $\{v_j(t)\}_{j=0}^{\infty}$ is uniformly convergent on $[t_0 - h, T]$. Let $v(t) = \lim_{n\to\infty} v_n(t)$. It is clear that the function $v(t)$ is continuous and it is a solution of the problem without impulses (3.1), (3.3) on the interval $[t_0 - h, T]$. Indeed, from the equality $v_n = \Omega(t_0, \varphi, v_{n-1})$ after taking a limit for $n \to \infty$ we obtain $v = \Omega(t_0, \varphi, v)$. This implies that the function $v(t)$ is a solution of the initial value problem for the differential-difference equation without impulses (3.4), (3.5). Then the inequality

$$v(t) \geq v_n(t), \quad t_0 - h \leq t \leq T$$

holds.

From equality $\tau_{v_n} = \tau(v_n(\tau_{v_n}))$ as $n \to \infty$ we obtain $\tau_v = \tau(v_n(\tau_v))$. Therefore the integral curve of the solution $v(t)$ of the initial value problem for the differential-difference equation without impulses (3.4), (3.5) intersects the curve σ at the moment τ_v. According to Lemma 3.1.3 this moment is unique.

Since $\tau_{v_n} \le \tau_v$, it is clear that the function v_n^+ is defined for $\tau_v \le t \le T$, $n = 1, 2, \ldots$. According to (3.20) the sequence of functions $\{v_j^+(t)\}_{j=1}^{\infty}$ is a monotone increasing in their common domain. We will prove that the sequence of functions $\{v_j^+(t)\}_{j=1}^{\infty}$ is bounded from above. Indeed, for $t \in (\tau_{v_n} - h, \tau_{v_n})$ the equalities

$$v_n^+(t) = v_n(t) \le M_v + M(T - t_0), \quad n = 1, 2, \ldots$$

hold.

Then for $t \in [\tau_{v_n}, T]$ the inequalities are valid

$$\begin{aligned}
v_n^+(t) &= v_n(\tau_{v_n}) + I(v_n(\tau_{v_n})) + \int_{\tau_{v_n}}^{t} f(s, v_{n-1}^+(s), v_{n-1}^+(s-h))ds \\
&\quad + \int_{\tau_{v_n}}^{t} \Big(L_1(v_{n-1}^+(s) - v_n^+(s)) + L_2(v_{n-1}^+(s-h) - v_n^+(s-h))\Big)ds \\
&\le M_v + \int_{\tau_{v_n}}^{t} f(s, v_{n-1}^+(s), v_{n-1}^+(s-h))ds \le M_v + M(t - t_{v_n}) \\
&\le M_v + M(T - \tau_{v_n}) = M_v^+. \qquad (3.22)
\end{aligned}$$

Therefore the sequence $\{v_j^+(t)\}_{j=1}^{\infty}$ is uniformly convergent for $\tau_v \le t \le T$. Let $v^+(t) = \lim_{n\to\infty} v_n^+(t)$. The inequalities

$$v^+(t) \ge v_n^+(t), \quad \tau_v \le t \le T, \quad n = 1, 2, \ldots$$

hold.

From equality $v_n^+(\tau_{v_n}) = v_n(\tau_{v_n}) + I(v_n(\tau_{v_n}))$ as $n \to \infty$ we obtain

$$v^+(\tau_v) = v(\tau_v) + I(v(\tau_v)).$$

From the equality $v_n^+ = \Omega(\tau_{v_n}, \varphi_1, v_{n-1}^+)$ we conclude that the function $v^+(t)$ satisfies the equation

$$\frac{dv^+(t)}{dt} = \Omega(\tau_v, \varphi_1, v^+) = f(t, v^+(t), v^+(t-h))$$

with initial condition

$$\varphi_1(t) = \begin{cases} v(t) & \text{for } \tau_v - h \le t < \tau_v \\ v(t) + I(v(t)) & \text{for } t = \tau_v. \end{cases}$$

Set

$$u(t) = \begin{cases} v(t) & \text{for } t_0 - h \le t < \tau_v \\ v^+(t) & \text{for } \tau_v \le t \le T. \end{cases}$$

The function $u(t)$ is a solution of the initial value problem for the impulsive differential-difference equation (3.1), (3.2), (3.3). □

In the case when the sequence of upper solutions is used as list of approximations of the solution we obtain the following result:

Theorem 3.1.2. *Let the following conditions be fulfilled:*

1. *Conditions* H1–H6 , A2, A3, *and* A4 *are satisfied.*

2. *Function* $\tau(x)$ *is decreasing on* $\mathbf{R}$.

There exists a sequence of functions $\{u_n(t)\}_0^\infty$ *such that* $u_n(t) : [t_0 - h, T] \to \mathbf{R}$ *and:*

a/ Functions $u_n(t)$ *are upper solutions of the initial value problem for the impulsive differential-difference equation* (3.1), (3.2), (3.3)*;*

b/ The sequence of functions $\{u_n(t)\}_0^\infty$ *is nonincreasing, i.e.*

$$u_1(t) \geq u_2(t) \geq u_3(t) \geq \dots \quad \text{for } t_0 - h \leq t \leq T;$$

c/ There exists the limit $u(t) = \lim_{n\to\infty} u_n(t)$ *for* $t_0 - h \leq t \leq T$*;*

d/ Function $u(t)$ *is a solution of the initial value problem for the impulsive differential-difference equation* (3.1), (3.2), (3.3) *on the interval* $[t_0 - h, T]$.

The proof of Theorem 3.1.2 is analogous to the proof of Theorem 3.1.1. □

Remark 7. If the delay of the argument is zero in (3.1), (3.2), i.e. $h = 0$, then this equation reduces to the impulsive ordinary differential equation with variable moments of impulses. The proved Theorems 1.1.1 and 1.1.2 give us procedures for approximately solving the initial value problem for the scalar impulsive differential equation with variable moments of impulses, that is studied in [81].

Remark 8. If the impulsive function $I(x) \equiv 0$ in (3.1), (3.2), then the obtained results give us procedures for approximately solving the initial value problem for the scalar differential-difference equation without impulses.

Remark 9. If the delay of argument $h = 0$ and the impulsive function $I(x) \equiv 0$ in (3.1), (3.2), then the proved results give us procedures for approximately solving the initial value problem for a scalar ordinary differential equation without impulses.

3.2. Monotone-Iterative Techniques for the Initial Value Problem for Systems of Impulsive Functional-Differential Equations

We will study the case when the monotone-iterative technique is applied to a system of impulsive differential-difference equations with fixed moments of impulses. We will define different types of lower and upper solutions of the studied impulsive system. We will give an algorithm for constructing two sequences of successive approximations of the solution of the considered problem.

We note that some qualitative results for impulsive functional-difference equations are obtained in [**?**]–[**?**], [**?**].

Let points $t_0 = 0 < t_1 < t_2 < \dots < t_p < T = t_{p+1}$ be fixed.

Consider the initial value problem for the system of impulsive functional-differential equations

$$x' = f(t,x(t),x(h(t))), \text{ for } t \in [0,T],\ t \neq t_i, \tag{3.23}$$

$$x(t_i+0) - x(t_i-0) = I_i(x(t_i)), \quad i = 1,2,\dots,p \tag{3.24}$$

$$x(t) = \varphi(t), \quad t \in [-a,0], \tag{3.25}$$

where $x \in \mathbf{R}^n$, $f : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $I_i :\ \mathbf{R}^n \to \mathbf{R}^n$, $h : [0,T] \to \mathbf{R}$, $T, a = const > 0$.

We will use notations that are analogous to those used in [92] for systems of ordinary differential equations. These notations play an important role in the definitions of different types of lower and upper solutions for the systems of impulsive equations.

For each natural number $j :\ 1 \leq j \leq n$ we consider two nonnegative integers p_j and q_j such that $p_j + q_j = n-1$ and for the points $x, y, z \in \mathbf{R}^n$ we introduce the notation

$$(x_j,[z]_{p_j},[y]_{q_j}) = \begin{cases} (z_1,\dots,z_{j-1},x_j,z_{j+1},\dots,z_{p_j+1},y_{p_j+2},\dots,y_n) \\ \qquad \text{for } p_j > j \\ (z_1,z_2,\dots,z_{p_j},y_{p_j+1},\dots,y_{j-1},x_j,y_{j+1},\dots,y_n) \\ \qquad \text{for } p_j \leq j. \end{cases} \tag{3.26}$$

For example, let $n = 3$. Choose $p_1 = 2, q_1 = 0$, $p_2 = 1$, $q_2 = 1$ and $p_3 = 1$, $q_3 = 1$. Then $(x_1,[z]_{p_1},[y]_{q_1}) = (x_1,z_2,z_3)$, $(x_2,[z]_{p_2},[y]_{q_2}) = (z_1,x_2,y_3)$, $(x_3,[z]_{p_3},[y]_{q_3}) = (z_1,y_2,x_3)$.

According to the introduced notation the initial value problem (3.23), (3.24), (3.25) can be rewritten in the form

$$x_j' = f_j(t,x_j(t),[x(t)]_{p_j},[x(t)]_{q_j},x_j(h(t)),[x(h(t))]_{p_j},[x(h(t))]_{q_j}), \quad t \neq t_i, \tag{3.27}$$

$$x_j(t_i+0) - x_j(t_i-0) = I_{ij}(x_j(t_i-0),[x(t_i-0)]_{p_j},[x(t_i-0)]_{q_j}), \ i = 1,2,\dots,p, \tag{3.28}$$

$$x_j(t) = \varphi_j(t), \quad t \in [-a,0], \quad j = 1,2,\dots,n. \tag{3.29}$$

Let $x = (x_1,x_2,\dots,x_n)$ and $y = (y_1,y_2,\dots,y_n)$. We will say that the inequality $x \leq (\geq) y$ holds, if for all natural numbers $j : 1 \leq j \leq n$ the inequalities $x_j \leq (\geq) y_j$ hold.

Definition 20. The pair of functions $v, w \in PC([-a,T],\mathbf{R}^n)$, $v = (v_1,v_2,\dots,v_n)$, $w = (w_1,w_2,\dots,w_n)$ is called *a pair of lower and upper quasisolutions* of the initial value problem for the system of impulsive functional-differential equations (3.23), (3.24), (3.25) if

$$v_j' \leq f_j(t,v_j(t),[v(t)]_{p_j},[w(t)]_{q_j},v_j(h(t)),[v(h(t))]_{p_j},[w(h(t))]_{q_j}), \tag{3.30}$$

$$w_j' \quad \geq \quad f_j(t,w_j(t),[w(t)]_{p_j},[v(t)]_{q_j},w_j(h(t)),[w(h(t))]_{p_j},[v(h(t))]_{q_j}) \\ \text{for } t \in [0,T],\ t \neq t_i,$$

$$v_j(t_i+0) - v_j(t_i-0) \leq I_{ij}(v_j(t_i-0),[v(t_i-0)]_{p_j},[w(t_i-0)]_{q_j}), \tag{3.31}$$

$$w_j(t_i+0) - w_j(t_i-0) \quad \geq \quad I_{ij}(w_j(t_i-0),[w(t_i-0)]_{p_j},[v(t_i-0)]_{q_j}), \\ i = 1,2,\dots,p,$$

$$v_j(t) \leq \varphi_j(t), \qquad w_j(t) \geq \varphi_j(t), \quad t \in [-a,0], \quad j = 1,2,\dots,n. \tag{3.32}$$

Remark 10. We will note that the pair of lower and upper quasisolutions is generalization of the lower and upper solutions in the scalar case ($n = 1, p_1 = q_1 = 0$).

Definition 21. The pair of functions $v, w \in PC([-a,T], \mathbf{R}^n)$, $v = (v_1, v_2, \dots, v_n)$, $w = (w_1, w_2, \dots, w_n)$ is called *a pair of quasisolutions* of the initial value problem for the system of impulsive functional-differential equations (3.23), (3.24), (3.25) if (3.30), (3.31), (3.32) are satisfied only for equalities.

Definition 22. The pair of functions $v, w \in PC([-a,T], \mathbf{R}^n)$, $v = (v_1, v_2, \dots, v_n)$, $w = (w_1, w_2, \dots, w_n)$ is called *a pair of minimal and maximal quasisolutions* of the initial value problem for the system of impulsive functional-differential equations (3.23), (3.24), (3.25) if it is a pair of quasisolutions of the same problem, $v(t) \le w(t)$ and for any other pair (μ, ν) of quasisolutions of (3.23), (3.24), (3.25), the inequalities $v(t) \le \mu(t) \le w(t)$, $v(t) \le \nu(t) \le w(t)$ hold for $t \in [-a,T]$.

Remark 11. We will note that the pair of the minimal and maximal quasisolutions is generalization of the minimal and maximal solutions in the scalar case ($n = 1, p_1 = q_1 = 0$).

Remark 12. We will note that if the pair of functions $v, w \in PC([-a,T], \mathbf{R}^n)$ is a pair of minimal and maximal quasisolutions, then the inequality $v(t) \le w(t)$ holds. Also, for any pair of quasisolutions this inequality is false.

Remark 13. We will note that for all natural numbers $j: \ 1 \le j \le n$ the equalities $p_j = n-1$ and $q_j = 0$ hold and the pair $v, w \in PC([-a,T], \mathbf{R}^n)$ is a pair of quasisolutions of (3.23), (3.24), (3.25). Then the functions v and w are solutions of the same problem. If the initial value problem (3.23), (3.24), (3.25) has unique solution $u(t)$, then the pair of minimal and maximal quasisolutions is (u,u).

For all pairs of functions $v, w \in PC([-a,T], \mathbf{R}^n$ such that $v(t) \le w(t)$ for $t \in [-a,T]$, we define the sets

$$S(v,w) = \{u \in PC([-a,T], \mathbf{R}^n): \ \ v(t) \le u(t) \le w(t), t \in [-a,T]\}, \tag{3.33}$$

$$\Gamma_i(v,w) = \{x \in \mathbf{R}^n, v(t_i) \le x_i \le w(t_i)\}, \ \ i = 1,2\dots,p. \tag{3.34}$$

Lemma 3.2.1. *Let the following conditions be fulfilled:*

1. *Function $h \in C([0,T], \mathbf{R})$ is nondecreasing and it satisfies the inequalities $-a \le h(t) \le t$ for $t \in [0,T]$.*

2. *The scalar function $m \in PC^1([-a,T], \mathbf{R})$ satisfies the inequalities*

$$m'(t) \le -Mm(t) - Nm(h(t))), \ \ for \ \ t \in [0,T] \ \ t \ne t_i, \tag{3.35}$$

$$m(t_i + 0) - m(t_i - 0) \le -L_i m(t_i), \quad i = 1,2,\dots,p, \tag{3.36}$$

$$m(t) = m(0), \quad t \in [-a,0], \tag{3.37}$$

$$m(0) \le 0, \tag{3.38}$$

where M and L are positive constants, $0 \le L_i < 1, \ \ i = 1,2,\dots,p$ *and*

$$(M+N)\tau < (1-L)^{p+1}, \tag{3.39}$$

$$\tau = \max\{t_{i+1} - t_i : \ i = 0, 1, 2, \ldots, p\}, \tag{3.40}$$

$$L = \max\{L_i : \ i = 1, 2, \ldots, p\}. \tag{3.41}$$

Then the inequality $m(t) \leq 0$ *holds for* $t \in [-a, T]$.

Proof. Assume the contrary, i.e. there exists a point $\xi \in (0, T]$ such that $m(\xi) > 0$. Consider the following three cases:

Case 1. Let $m(0) = 0,\ m(t) \geq 0, m(t) \not\equiv 0$ for $t \in [0, b]$, where $b \in (0, t_1)$ is a small enough constant. Then from equality (3.37) follows that $m(t) \equiv 0$ for $t \in [-a, 0]$. From inequality (3.36) follows that if for a natural number $k : \ 1 \leq k \leq p$ the equality $m(t_k) = 0$ holds, then $m(t_k + 0) \leq 0$. Therefore there exist points $\xi_1, \xi_2 \in [0, T]$, $\xi_1 < \xi_2$ such that $\xi_1, \xi_2 \in [0, t_1]$ or $\xi_1, \xi_2 \in (t_j, t_{j+1}]$ for a natural number $j : \ 1 \leq j \leq p$, and $m(t) = 0$ for $t \in [-a, \xi_1]$, and $m(t) > 0$ for $t \in (\xi_1, \xi_2)$. From inequality (3.35) follows that $m'(t) \leq 0$ for $t \in (\xi_1, \xi_2]$. Therefore, function $m(t)$ is continuous nonincreasing function on $[\xi_1, \xi_2]$, i.e. $m(t) \leq m(\xi_1) = 0$ for $t \in [\xi_1, \xi_2]$. The last inequality contradicts the assumption.

Case 2. Let $m(0) < 0$. According to the assumption and inequality (3.36) there exists a point $\eta \in (0, T], \eta \neq t_i,\ i = 1, 2, \ldots, p$ such that $m(\eta) \leq 0$ for $t \in [-a, \eta]$, $m(\eta) = 0$, $m(t) > 0$ for $t \in (\eta, \eta + \varepsilon)$, where $\varepsilon > 0$ is a small enough constant. Denote

$$\inf\{m(t) : \ t \in [-a, \eta]\} = -\lambda < 0.$$

Consider the following two cases:

Case 2.1. Let there exists a point $\varsigma \in [-a, \eta], \varsigma \neq t_i, i = 1, 2, \ldots, p$ such that $m(\varsigma) = -\lambda$ exist. Assume that $\varsigma \in (t_k, t_{k+1}]$ and $\eta \in (t_{k+s}, t_{k+s+1}]$ for some integers $k, s : \ 0 \leq k \leq p,\ 0 \leq s \leq p - k$. Choose a point $\eta_1 \in (t_{k+s}, t_{k+s+1}], \eta_1 > \eta$ such that $g(\eta_1) > 0$. According to the mean value theorem the equalities

$$\begin{aligned}
&m(\eta_1) - m(t_{k+s} + 0) = m'(\xi_s)(\eta_1 - t_{k+s}),\\
&m(t_{k+s} - 0) - m(t_{k+s-1} + 0) = m'(\xi_{s-1})(t_{k+s} - t_{k+s-1}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$m(t_{k+1} - 0) - m(\varsigma) = m'(\xi_0)(t_{k+1} - \varsigma), \tag{3.42}$$

hold, where $\xi_0 \in (\varsigma, t_{k+1})$, $\xi_s \in (t_{k+s}, \eta_1)$, $\xi_i \in (t_{k+i}, t_{k+i+1})$, $i = 1, 2, \ldots, s - 1$.

From inequality (3.36) and equalities (3.42) we obtain the inequalities

$$\begin{aligned}
&m(\eta_1) - (1 - L_{k+s})m(t_{k+s}) \leq m'(\xi_s)\tau,\\
&m(t_{k+s}) - (1 - L_{k+s-1})m(t_{k+s-1}) \leq m'(\xi_{s-1})\tau,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$m(t_{k+1}) - m(\varsigma) \leq m'(\xi_0)\tau, \tag{3.43}$$

From inequalities (3.43) we obtain the following inequality

$$\begin{aligned}
&m(\eta_1) - (1 - L_{k+1})(1 - L_{k+2}) \ldots (1 - L_{k+s})m(\varsigma)\\
&\leq \Big[m'(\xi_s) + (1 - L_{k+s})m'(\xi_{s-1}) + (1 - L_{k+s})(1 - L_{k+s-1})m'(\xi_{s-2})\\
&+ \cdots + (1 - L_{k+s})(1 - L_{k+s-1}) \ldots (1 - L_{k+1}m'(\xi_0)\Big]\tau.
\end{aligned}$$

From the above inequality, inequality (3.35) and the choice of points η_1 and ς follows the inequality

$$(1-L)^s\lambda \leq \Big[1+(1-l)+(1-l)^2+\cdots+(1-l)^s\Big](M+N)\lambda\tau.$$

Therefore, inequality

$$(1-L)^s \leq \frac{(M+N)\tau}{1-l} \tag{3.44}$$

holds.

Inequality (3.44) contradicts inequality (3.39).

Case 2.2. Let there exist a point $t_k \in [0,\eta)$ such that $m(t_k+0) > m(t)$ for $t \in [0,\eta)$, i.e. $m(t_k+0) = -\lambda$. As in the proof of case 2.1, where $\varsigma = t_k+0$, we obtain a contradiction.

Case 3. Let $m(0) = 0$, $m(t) \leq 0$, and $m(t) \not\equiv 0$ for $t \in [0,b]$, where $b > 0$ is a small enough constant. As in the proof of case 2, we obtain a contradiction, that proves Lemma 3.2.1. □

We will give an algorithm for constructing a sequence of successive approximations and we will prove the application of monotone iterative technique to the initial value problem for a system of nonlinear impulsive functional-differential equations.

Theorem 3.2.1. *Let the following conditions be fulfilled:*

1. *The pair of functions $v,w \in PC([0,T],\mathbf{R}^n)$, $v = (v_1,v_2,\dots,v_n)$, $w = (w_1,w_2,\dots,w_n)$ is a pair of lower and upper quasisolutions of the initial value problem* (3.23), (3.24), (3.25), *$v(t) \leq w(t)$ for $t \in [-a,T]$, and $v(0)-\varphi(0) \leq v(t)-\varphi(t)$, $w(0)-\varphi(0) \geq w(t)-\varphi(t)$ for $t \in [-a,0]$.*

2. *Function $f : [0,T]\times\mathbf{R}^n\times\mathbf{R}^n \to \mathbf{R}^n$, $f = (f_1,f_2,\dots,f_n)$, where $f_j(t,x,y) = f_j(t,x_j,[x]_{p_j},[x]_{q_j},y_j,[y]_{p_j},[y]_{q_j})$, is nondecreasing in $[x]_{p_j}$ and $[y]_{p_j}$, nonincreasing in $[x]_{q_j}$ and $[y]_{q_j}$, and for $x,y \in S(v,w)$, $y(t) \leq x(t)$ the inequality*

$$\begin{aligned}
&f_j(t,x_j(t),[x(t)]_{p_j},[x(t)]_{q_j},x_j(h(t)),[x(h(t))]_{p_j},[x(h(t))]_{q_j})\\
&\quad -f_j(t,y_j(t),[y(t)]_{p_j},[y(t)]_{q_j},y_j(h(t)),[y(h(t))]_{p_j},[y(h(t))]_{q_j})\\
&\geq -M_j(x_j(t)-y_j(t))-N_j(x_j(h(t))-y_j(h(t))),\\
&\quad t\in[0,T],\quad j=1,2,\dots,n
\end{aligned} \tag{3.45}$$

holds, where M_j,N_j, $j=1,2,\dots,n$ are positive constants.

3. *Functions $I_i : \mathbf{R}^n \to \mathbf{R}^n$, where $I_i = (I_{i1},I_{i2},\dots,I_{in})$ and the functions $I_{ij}(x) = I_{ij}(x_j,[x]_{p_j},[x]_{q_j})$ are nondecreasing in $[x]_{p_j}$, nonincreasing in $[x]_{q_j}$ and for $x,y \in \Gamma_i(v,w)$, $y \leq x$ the inequality*

$$I_{ij}(x_j,[x]_{p_j},[x]_{q_j}) - I_{ij}(y_j,[y]_{p_j},[y]_{q_j}) \geq -L_{ij}(x_j-y_j),\quad j=1,2,\dots,p$$

holds, where $L_{ij} :\ 0 < L_{ij} < 1$, $i=1,2,\dots,n$, $j=1,2,\dots p$ are constants.

4. *Function $h \in C([0,T],\mathbf{R})$ is nondecreasing and the inequalities $-a \leq h(t) \leq t$ hold for $t \in [0,T]$.*

5. *The inequalities*

$$(M_j+N_j)\tau < (1-l_j)^{p+1},$$
$$\tau = \max\{t_{i+1}-t_i;\ i=0,1,2,\dots,p\},$$
$$l_j = \max\{L_{ij}:\ i=1,2,\dots,p\} \tag{3.46}$$

hold.

Then there exist two sequences of functions $\{v^{(k)}(t)\}_0^\infty$ *and* $\{w^{(k)}(t)\}_0^\infty$ *such that:*

a/ The sequences are increasing and decreasing correspondingly;

b/ The pair of functions $v^{(k)}(t)$, $w^{(k)}(t)$ *is a pair of lower and upper quasisolutions of the initial value problem for the system of nonlinear impulsive functional-differential equations* (3.23), (3.24), (3.25)*;*

c/ Both sequences converge on $[-a,T]$*;*

d/ The limits $V(t) = \lim_{k\to\infty} v^{(k)}(t)$, $W(t) = \lim_{k\to\infty} w^{(k}(t)$ *are a pair of minimal and maximal solutions of the initial value problem for the system of nonlinear impulsive functional-differential equations* (3.23), (3.24), (3.25).

e/ If $u(t) \in S(v,w)$ *is a solution of the initial value problem for the system of nonlinear impulsive functional-differential equations* (3.23), (3.24), (3.25)*, then* $V(t) \le u(t) \le W(t)$.

Proof. We fix two arbitrary functions $\eta,\mu \in S(v,w)$ and for all natural numbers $j:\ 1\le j\le n$ we consider the initial value problem for the scalar linear impulsive functional-differential equation

$$u'(t)+M_ju(t)+N_ju(h(t)) = \psi_j(t,\eta,\mu),\ \text{ for } t\in[0,T]\ \ t\neq t_i, \tag{3.47}$$

$$u(t_i+0)-u(t_i-0) = -L_{ij}u(t_i)+\gamma_{ij}(\eta,\mu),\quad i=1,2,\dots,p, \tag{3.48}$$

$$u(t) = \varphi_j(t),\quad t\in[-a,0], \tag{3.49}$$

where $u \in \mathbf{R}$,

$$\begin{aligned}\psi_j(t,\eta,\mu) &= f_j(t,\eta_j(t),[\eta(t)]_{p_j},[\mu(t)]_{q_j},\eta_j(h(t)),[\eta(h(t))]_{p_j},[\mu(h(t))]_{q_j})\\ &\qquad +M_j\eta_j(t)+N_j\eta_j(h(t)),\\ \gamma_{ij}(\eta,\mu) &= I_{ij}(\eta_j(t_i),[\eta(t_i)]_{p_j},[\mu(t_i)]_{q_j})+L_{ij}\eta_j(t_i),\ \ i=1,2,\dots,p\end{aligned}$$

The initial value problem (3.47)–(3.49) has an unique solution for the fixed pair of functions $\eta,\mu\in S(v,w)$.

For any two functions $\eta,\mu\in S(v,w)$ such that $\eta(t)\le\mu(t)$ for $t\in[-a,T]$ we define operator $\Omega: S(v,w)\times S(v,w)\to S(v,w)$ by $\Omega(\eta,\mu) = x(t)$, where $x(t) = (x_1(t),x_2(t),\dots,x_n(t))$ and $x_j(t)$ is the unique solution of the initial value problem for the scalar impulsive equation (3.47)–(3.49) for the pair of functions η,μ .

We will proof inequality $v\le\Omega(v,w)$. We denote $m(t) = v(t)-v^{(1)}(t)$, where $v^{(1)}(t) = \Omega(v,w)$. The function $m(t)$ satisfies the inequalities (3.35)- (3.38). According to Lemma 3.2.1 the function $m(t)$ is nonpositive, i.e. $v\le\Omega(v,w)$. Analogously it can be proved that the inequality $w\ge\Omega(w,v)$ holds.

Let $\eta,\mu\in S(v,w)$ be arbitrary functions such that $\eta(t)\le\mu(t)$ for $t\in[-a,T]$. We introduce the notations $x^{(1)} = \Omega(\eta,\mu)$, $x^{(2)} = \Omega(\mu,\eta)$, $g = x^{(1)}-x^{(2)}$, $g = (g_1,g_2,\dots,g_n)$. According to Lemma 3.2.1 functions $g_j(t)$ are nonpositive, i.e. $\Omega(\eta,\mu)\le\Omega(\mu,\eta)$.

We define the sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)} = v, \qquad w^{(0)} = w,$$
$$v^{(k+1)} = \Omega(v^{(k)}, w^{(k)}), \qquad w^{(k+1)} = \Omega(w^{(k)}, v^{(k)}).$$

According to the above proofs, functions $v^{(k)}(t)$ and $w^{(k)}(t)$ satisfy for $t \in [-a,T]$ the following inequalities

$$v^{(0)}(t) \le v^{(1)}(t) \le \cdots \le v^{(k)}(t) \le w^{(k)}(t) \le \cdots \le w^{(1)}(t) \le w^{(0)}(t). \tag{3.50}$$

Both sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are convergent on $[-a,T]$. Let $V_j(t) = \lim_{k\to\infty} v_j^{(k)}(t)$, $W_j(t) = \lim_{k\to\infty} w_j^{(k)}(t)$, $j = 1,2,\dots,n$. We will prove that the pair of functions $V(t)$ and $W(t)$, where $V = (V_1,V_2,\dots,V_n)$ and $W = (W_1,W_2,\dots,W_n)$, are a pair of minimal and maximal quasisolutions of the initial value problem (3.23), (3.24), (3.25). From the definition of functions $v^{(k)}(t)$, $v^{(k)} = (v_1^{(k)}, v_2^{(k)},\dots,v_n^{(k)})$ and $w^{(k)}(t)$, $w^{(k)} = (w_1^{(k)}, w_2^{(k)},\dots,w_n^{(k)})$ follows that these functions satisfy the initial value problem ($j = 1,2,\dots,n$)

$$(v_j^{(k)}(t))' + M_j v_j^{(k)}(t) + N_j v_j^{(k)}(h(t))) = \psi_j(t, v^{(k-1)}, w^{(k-1)}), \tag{3.51}$$

$$(w_j^{(k)}(t))' + M_j w_j^{(k)}(t) + N_j w_j^{(k)}(h(t))) = \psi_j(t, w^{(k-1)}, v^{(k-1)}),$$
$$\text{for } t \in [0,T] \;\; t \ne t_i,$$

$$v_j^{(k)}(t_i+0) - v_j^{(k)}(t_i-0) = -L_{ij} v_j^{(k)}(t_i) + \gamma_{ij}(v^{(k-1)}, w^{(k-1)}), \tag{3.52}$$

$$w_j^{(k)}(t_i+0) - w_j^{(k)}(t_i-0) = -L_{ij} w_j^{(k)}(t_i) + \gamma_{ij}(w^{(k-1)}, v^{(k-1)}),$$
$$i = 1,2,\dots,p,$$

$$v_j^{(k)}(t) = v_j^{(k)}(0), \quad w_j^{(k)}(t) = w_j^{(k)}(0), \quad t \in [-a,0]. \tag{3.53}$$

From equations (3.51)–(3.53) follows that the pair of functions $V(t)$ and $W(t)$ is a pair of quasisolutions of the initial value problem (3.23), (3.24), (3.25). Let $u,z \in S(v,w)$ be a pair of quasisolutions of the initial value problem (3.23), (3.24), (3.25). From inequalities (3.50) follows that there exists a natural number k such that $v^{(k)}(t) \le u(t) \le w^{(k)}(t)$ and $v^{(k)}(t) \le z(t) \le w^{(k)}(t)$ for $t \in [-a,T]$. We introduce the notation $g(t) = v^{(k+1)}(t) - u(t)$, $g = (g_1, g_2,\dots,g_n)$. According to Lemma 3.2.1 the inequalities $g_j(t) \le 0$, $j = 1,2,\dots$ hold for $t \in [-a,T]$, i.e. $v^{(k+1)}(t) \le u(t)$.

Analogously the validity of inequalities $w^{(k+1)}(t) \ge u(t)$ and $v^{(k+1)}(t) \le z(t) \le w^{(k+1)}(t)$ for $t \in [-a,T]$ can be proved.

Let $u(t) \in S(v,w)$ be a solution of the initial value problem (3.23), (3.24), (3.25). Consider the pair of functions (u,u) that is a pair of quasisolutions of the initial value problem (3.23), (3.24), (3.25). According to the proof given above the inequality $V(t) \le u(t) \le W(t)$ holds for $t \in [-a,T]$. □

Remark 14. As a partial case of the proved results we obtain the monotone iterative techniques for

– the initial value problem for systems of impulsive differential-difference equations with fixed moments of impulses, where $h(t) = t - h$, $h > 0$ is a constant ([69]);

– the initial value problem for systems of impulsive differential equations with fixed moments of impulses ($h(t) \equiv 0$);

– the initial value problem for scalar nonlinear impulsive differential-difference equations with fixed moments of impulses ($n = 1$). In this case differently than the previous section the procedure gives us the possibility of constructing two sequences of successive approximations (increasing and decreasing sequences);

– the initial value problem for a scalar nonlinear impulsive differential equation with fixed moments of impulses;

– the initial value problem for differential equations as well as differential-difference equations without impulses (scalar and n-dimensional cases) ([71], ([88], [92], [94]).

3.3. Monotone-Iterative Techniques for Periodic Boundary Value Problem for Systems of Impulsive Differential-Difference Equations

The boundary value problems for various types of differential equations have been investigated by many authors ([1], [82], [99]).

In this section we will consider a periodic boundary value problem for systems of impulsive differential-difference equations with fixed moments of impulses. We will apply the monotone-iterative technique for approximately solving of this type of problems. For this purpose, we will define various types of pairs of quasisolutions and pairs of lower and upper quasisolutions.

Let points $t_0 = 0 < t_1 < t_2 < \cdots < t_p < T = t_{p+1}$ be fixed.

Consider the periodic boundary value problem for systems of impulsive differential-difference equations with fixed moments of impulses

$$x' = f(t, x(t), x(t-h)) \quad \text{for } t \in [0, T] \;\; t \neq t_i, \tag{3.54}$$

$$x(t_i + 0) - x(t_i - 0) = I_i(x(t_i)), \quad i = 1, 2, \ldots, p, \tag{3.55}$$

$$x(t) = x(0) \quad \text{for } t \in [-h, 0], \tag{3.56}$$

$$x(0) = x(T), \tag{3.57}$$

where $x \in \mathbf{R}^n$, $f : [t_0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $I_i :\ \mathbf{R}^n \to \mathbf{R}^n$, $h = const > 0$, $T = const > 0$.

According to the introduced in the previous section notation (3.24), the periodic boundary value problem (3.54)–(3.57) can be written in the form

$$x_j' = f_j(t, x_j(t), [x(t)]_{p_j}, [x(t)]_{q_j}, x_j(t-h), [x(t-h)]_{p_j}, [x(t-h)]_{q_j}), \quad t \neq t_i, \tag{3.58}$$

$$x_j(t_i + 0) - x_j(t_i - 0) = I_{ij}(x_j(t_i - 0), [x(t_i - 0)]_{p_j}, [x(t_i - 0)]_{q_j}), \;\; i = 1, 2, \ldots, p, \tag{3.59}$$

$$x_j(t) = x_j(0) \quad \text{for } t \in [-h, 0], \tag{3.60}$$

$$x_j(0) = x_j(T), \quad j = 1,2,\ldots,n. \tag{3.61}$$

Below is a list of the main definitions used in this section.

Definition 23. The pair of functions $v, w \in PC([0,T], \mathbf{R}^n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ is called *a pair of lower and upper quasisolutions* of the periodic boundary value problem for the system of impulsive differential-difference equations with fixed moments of impulses (3.54)–(3.57) if

$$v'_j \leq f_j(t, v_j(t), [v(t)]_{p_j}, [w(t)]_{q_j}, v_j(t-h), [v(t-h)]_{p_j}, [w(t-h)]_{q_j}), \tag{3.62}$$

$$w'_j \geq f_j(t, w_j(t), [w(t)]_{p_j}, [v(t)]_{q_j}, w_j(t-h), [w(t-h)]_{p_j}, [v(t-h)]_{q_j}) \quad \text{for } t \in [0,T] \ \ t \neq t_i,$$

$$v_j(t_i+0) - v_j(t_i-0) \leq I_{ij}(v_j(t_i-0), [v(t_i-0)]_{p_j}, [w(t_i-0)]_{q_j}), \tag{3.63}$$

$$w_j(t_i+0) - w_j(t_i-0) \geq I_{ij}(w_j(t_i-0), [w(t_i-0)]_{p_j}, [v(t_i-0)]_{q_j}), \quad i = 1,2,\ldots,p,$$

$$v_j(t) = v_j(0), \quad w_j(t) = w_j(0) \quad \text{for } t \in [-h,0], \tag{3.64}$$

$$v_j(0) \leq v_j(T), \quad w_j(0) \geq w_j(T) \ \ j = 1,2,\ldots,n. \tag{3.65}$$

Remark 15. It is worth mentationing that the introduced definition for a pair of lower and upper quasisolutions is generalization of the lower and upper solutions in the scalar case ($n = 1, p_1 = q_1 = 0$).

Definition 24. The pair of functions $v, w \in PC([0,T], \mathbf{R}^n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ is called *a pair of quasisolutions* of the periodic boundary value problem for the system of impulsive differential-difference equations with fixed moments of impulses (3.54)–(3.57) if the relations (3.62)–(3.65) hold only for equalities.

Definition 25. The pair of functions $v, w \in PC([0,T], \mathbf{R}^n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ is called *a pair of minimal and maximal quasisolutions* of the periodic boundary value problem for the system of impulsive differential- difference equations with fixed moments of impulses (3.54)–(3.57) if it is a pair of quasisolutions of the same problem, $v(t) \leq w(t)$ and for any other pair (μ, ν) of quasisolutions of (3.54)–(3.57), the inequalities $v(t) \leq \mu(t) \leq w(t)$, $v(t) \leq \nu(t) \leq w(t)$ hold for $t \in [0,T]$.

Remark 16. The defined above pair of minimal and maximal solution is generalization of the minimal and maximal solutions in the scalar case ($n = 1, p_1 = q_1 = 0$).

Remark 17. If the pair of functions $v, w \in PC([0,T], \mathbf{R}^n)$ is a pair of minimal and maximal solutions, then the inequality $v(t) \leq w(t)$ holds. This inequality could not be true for an arbitrary pair of quasisolutions.

Remark 18. If for all natural numbers $j: \ 1 \leq j \leq n$ the equalities $p_j = n-1$ and $q_j = 0$ hold and the pair $v, w \in PC([0,T], \mathbf{R}^n)$ is a pair of quasisolutions of the problem (3.54)–(3.57), then the functions v and w are solutions of the same problem. If the periodic boundary value problem (3.54)–(3.57) has unique solution $u(t)$, then the pair of minimal and maximal quasisolution is (u,u).

For all pairs of functions $v,w \in PC([0,T],\mathbf{R}^n)$ such that $v(t) \leq w(t)$ for $t \in [-h,T]$, we define the sets $S(v,w)$ and $\Gamma_i(v,w)$ with the help of equalities (3.33) and (3.34), where constant a is substituted by h.

We will prove the validity of some results for linear scalar impulsive differential-difference equations and inequalities.

Lemma 3.3.1. *Let $m \in PC^1([-h,T],\mathbf{R})$ satisfy the inequalities*

$$m'(t) \leq -Mm(t) - Nm(t-h)), \quad t \neq t_i, \ \ i = 1,2,\ldots,p, \tag{3.66}$$

$$m(t_i+0) - m(t_i-0) \leq -L_i m(t_i), \quad i = 1,2,\ldots,p, \tag{3.67}$$

$$m(t) = m(0), \quad t \in [-h,0], \tag{3.68}$$

$$m(0) \leq m(T), \tag{3.69}$$

where M and L are positive constants, $0 \leq L_i < 1, \ \ i = 1,2,\ldots,p$ *and*

$$(M+N)p\tau < (1-L)^p, \tag{3.70}$$

$$\tau = \max\{t_{i+1} - t_i : \ i = 0,1,2,\ldots,p\}, \tag{3.71}$$

$$L = \max\{L_i : \ i = 1,2,\ldots,p\}. \tag{3.72}$$

Then the inequality $m(t) \leq 0$ holds for $t \in [-h,T]$.

Proof. Assume the contrary, i.e. there exists a point $\xi \in [0,T]$ such that $m(\xi) > 0$. Consider the following three cases:

Case 1. Let $m(t) \geq 0, m(t) \not\equiv 0$ for $t \in [0,T]$. Then from equality (3.68) follows that $m(t) \geq 0$ for $t \in [-h,T]$. From inequality (3.66) we obtain the inequality $m'(t) \leq 0$ for $t \neq t_i, \ \ t \in [0,T]$. The last inequality together with inequality (3.67) prove that function $m(t)$ is nonincreasing on the interval $[0,T]$. Therefore for $t \in [0,T]$ the inequality

$$m(0) \geq m(t) \tag{3.73}$$

holds.

Inequalities (3.69) and (3.73) prove that $m(t) = c$ for $t \in [0,T]$, where $c = const > 0$. From inequality (3.66) follows the validity of the inequality

$$0 \leq (-M-N)c. \tag{3.74}$$

The obtained contradiction proves Lemma 3.3.1 in case 1.

Case 2. Let there exist a point $\eta \in [0,T]$ such that $m(\eta) < 0$ and $m(T) \geq 0$. We denote

$$\inf\{m(t) : \ t \in [-h,T]\} = -\lambda < 0.$$

Let us consider the following three possible cases:

Case 2.1. Let there exist a point $\varsigma \in [-h,T]$ such that $m(\varsigma) = -\lambda$. We assume that $\varsigma \in [0,T)$ and $\varsigma \in (t_k, t_{k+1}]$. According to the Mean Value Theorem the following equalities are satisfied:

$$\begin{aligned}
&m(T) - m(t_p + 0) = m'(\xi_0)(T - t_p),\\
&m(t_p - 0) - m(t_{p-1} + 0) = m'(\xi_1)(t_p - t_{p-1}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

$$m(t_{k+1} - 0) - m(\varsigma) = m'(\xi_{p-k})(t_{k+1} - \varsigma), \tag{3.75}$$

where $\xi_0 \in (t_p, T)$, $\xi_{p-k} \in (\varsigma, t_{k+1})$, $\xi_i \in (t_{p-i}, t_{p-i+1})$, $i = 1, 2, \dots, p-k-1$.

From inequality (3.67) and equalities (3.75) we obtain the inequalities

$$\begin{aligned}
&m(T) - (1 - L_p)m(t_p) \le m'(\xi_0)\tau,\\
&m(t_p) - (1 - L_{p-1})m(t_{p-1}) \le m'(\xi_1)\tau,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

$$m(t_{k+1}) - m(\varsigma) \le m'(\xi_{p-k})\tau, \tag{3.76}$$

From equalities (3.76) we obtain the inequality

$$\begin{aligned}
&m(T) - (1 - L_p)(1 - L_{p-1})\dots(1 - L_{k+1}m(\varsigma)\\
&\le \Big[m'(\xi_0) + (1 - L_p)m'(\xi_1) + (1 - L_p)(1 - L_{p-1})m'(\xi_2)\\
&+ \dots + (1 - L_p)(1 - L_{p-1})\dots(1 - L_{k+1}m'(\xi_{p-k})\Big]\tau.
\end{aligned}$$

From the above inequality, inequalities (3.66), and the choice of the point ς follows the validity of the inequality

$$\begin{aligned}
&(1 - L)^{p-k}\lambda \le \Big[1 + (1 - L_p) + (1 - L_p)(1 - L_{p-1}) + \dots\\
&+ (1 - L_p)(1 - L_{p-1})\dots(1 - L_{k+1}\Big](M + N)\lambda\tau,
\end{aligned}$$

or

$$(1 - L)^p \le (M + N)p\tau. \tag{3.77}$$

Inequality (3.77) contradicts inequality (3.70).

Case 2.2. Let there exist a point t_i, $i = 1, 2, \dots, p$ such that $m(t_i + 0) > m(t)$ for $t \in [0,T]$. Analogously to case 2.1, where $\varsigma = t_i + 0$, we obtain a contradiction.

Case 3. Let there exist a point $\eta \in [0,T]$, such that $m(\eta) < 0$ and $m(T) < 0$. Therefore, $m(t) < 0$ for $t \in [-h, 0]$. According to the assumptions there exists a point $\gamma \in [0,T]$ such that $m(\gamma) = 0$, $m(t) \le 0$ for $t \in [-h, \gamma]$ and $m(t) > 0$ for $t \in (\gamma, \gamma + \varepsilon)$, where $\varepsilon > 0$ is small enough number. We introduce the notation

$$\inf\{m(t) :\ t \in [-h, \gamma]\} = -\lambda < 0.$$

Analogously to the proof of case 2 we obtain a contradiction that proves Lemma 3.3.1.

□

Consider the periodic boundary value problem for the scalar linear impulsive differential-difference equation

$$u'(t)+Mu(t)+Nu(t-h))=\phi(t), \quad t\neq t_i, \;\; i=1,2,\dots,p, \tag{3.78}$$

$$u(t_i+0)-u(t_i-0)=-L_i u(t_i)+\gamma_i, \quad i=1,2,\dots,p, \tag{3.79}$$

$$u(t)=u(0), \quad t\in[-h,0], \tag{3.80}$$

$$u(0)=u(T), \tag{3.81}$$

where $u\in\mathbf{R}$.

We will prove an existence theorem for the periodic boundary value problem (3.78), (3.79), (3.80), (3.81).

Lemma 3.3.2. *Let the following conditions be fulfilled:*

1. *Function* $\phi\in C([0,T],\mathbf{R})$.

2. *Functions* $v,w\in PC([0,T],\mathbf{R})$ *are lower and upper solutions of the periodic boundary value problem* (3.78)–(3.81)*, and satisfy the inequality* $v(t)\leq w(t)$ *for* $t\in[-h,T]$.

3. *Inequality* (3.70) *holds, where* M *and* L *are positive constants,* $0\leq L_i<1$, $i=1,2,\dots,p$*, and the constants* τ *and* L *are defined by* (3.71) *and* (3.72).

Then the periodic boundary value problem (3.78)–(3.81) *has a solution* $x\in S(v,w)$.

Proof. To avoid some complicated notations and in order to be able to use the previous proofs, we will assume that $p=1$, i.e. there exists only one moment t_1 of impulse on the interval $[0,T]$.

Consider the scalar linear impulsive differential-difference equation (3.78), (3.79) with initial condition $x(t)=x(0)=x_0$ for $t\in[-h,0]$. Denote the solution of this problem by $x(t;x_0)$.

We will prove that there exists a point $x_0\in[v(0),w(0)]$ such that $x(0;x_0)=x(T;x_0)$.

Assume the contrary, i.e. for all points $x_0\in[v(0),w(0)]$ and all solutions $x(t;x_0)$ the inequality $x(0;x_0)\neq x(T;x_0)$ holds.

Case 1. Let $v(0)=w(0)$. Then $x_0=v(0)=w(0)$. Define $x(t;v(0))=v(t)$. From condition 2 follows that $v(0)\leq v(T)\leq w(T)\leq w(0)$ or $v(0)=x(0;v(0))=v(T)=x(T;v(0))$.

Case 2. Let $v(0)<w(0)$. Therefore, the following inequalities are satisfied:

$$x(0;v(0))<x(T;v(0)), \qquad x(0;w(0))>x(T;w(0)). \tag{3.82}$$

We will prove that there exists a number δ such that $0<\delta<w(0)-v(0)$ and for $0\leq w(0)-z<\delta$ the inequality $x(0;z)>x(T;z)$ holds. We assume the contrary, i.e. there exists a sequence of numbers $\{z_n\}_0^\infty$, $0\leq w(0)-z_n<\frac{1}{n}$ such that $x^{(n)}(0;z_n)<x^{(n)}(T;z_n)$.

Functions $x^{(n)}(t;z_n)$ satisfy the equalities

$$\begin{aligned}
x^{(n)}(t;z_n) &= z_n+\int_0^t f(s,x^{(n)}(s;z_n),x^{(n)}(s-h;z_n))ds, \quad t\in[0,t_1],\\
x^{(n)}(t;z_n) &= I_1(x^{(n)}(t;z_n))+\int_0^t f(s,x^{(n)}(s;z_n),x^{(n)}(s-h;z_n))ds,\\
&\qquad\qquad t\in(t_1,T],\\
x^{(n)}(t;z_n) &= z_n, \quad t\in[-h,0],
\end{aligned}$$

where $f(t,x(t),x(t-h)) = -Mx(t) - Nx(t-h) + \sigma(t)$, $I_1(x) = (1-L_1)x + \gamma_1$.

Therefore, there exists a subsequence $\{x^{(n_k)}(t;z_n)\}_0^\infty$ of the sequence of functions $\{x^{(n)}(t;z_n)\}_0^\infty$ that is uniformly convergent on the interval $[-h,T]$ and the limit $x(t)$ satisfies the relations

$$x(0) = w(0) \quad \text{and} \quad x(0) < x(T). \tag{3.83}$$

Therefore, the function $x(t)$ satisfies equations (3.78), (3.79) with initial condition $x(t) = x(0) = w(0)$ for $t \in [-h,0]$. Then from inequality (3.82) follows that the inequality

$$x(0) > x(T)$$

holds.

The last inequality contradicts inequality (3.83).

The obtained contradiction proves that there exists a number δ such that

$$0 < \delta < w(0) - v(0)$$

and for $0 \le w(0) - z < \delta$ the inequality

$$x(0;z) > x(T;z)$$

holds.

Consider the set Ψ of all numbers δ and let $\delta^* = \sup \Psi$.

Then

$$0 < \delta^* \le w(0) - v(0), \tag{3.84}$$

$$x(0;z) > x(T;z) \quad \text{for } 0 \le w(0) - z < \delta^*. \tag{3.85}$$

From the definition of number δ^* follows that there exists a sequence of numbers $\{z_n\}_0^\infty$ such that $v(0) < z_n < w(0) - \delta^*$, $\lim_{n\to\infty} z_n = w(0) - \delta^*$ and $x^{(n)}(0;z_n) > x(n)(T;z_n)$. From the conditions of Lemma 3.3.2 follows that there exists a subsequence $\{x^{(n_k)}(t;z_{n_k})\}_0^\infty$ of the sequence $\{x^{(n)}(t;z_n)\}_0^\infty$, that is uniformly convergent on $[-h,T]$.

Denote $x^*(t) = \lim_{n\to\infty} x^{(n_k)}(t;z_{n_k})$. From the definition of function $x^*(t)$ follows that the inequality

$$x^*(0) < x^*(T) \tag{3.86}$$

holds.

Function $x^*(t)$ satisfies equations (3.78), (3.79) with the initial condition $x^*(t) = x^*(0) = w(0) - \delta^*$ for $t \in [-h,0]$.

We choose a sequence of numbers $\{z_n\}_0^\infty$ such that

$$w(0) - \delta^* < z_n \le w(0), \quad \lim_{n\to\infty} z_n = w(0) - \delta^*.$$

For all number z_n we choose the solutions $x^{(n)}(t;z_n)$ of the linear scalar impulsive differential-difference equation (3.78), (3.79) with the initial condition $x^{(n)}(t;z_n) = z_n$ for $t \in [-h,0]$ such that

$$x^{(n)}(t;z_n) \ge x^*(t) \;\; \text{for} \;\; t \in [-h,0].$$

From the definition of number δ^* follows the validity of inequality $x^{(n)}(0;z_n) > x^{(n)}(T;z_n)$ for $n \geq 1$. From the conditions of Lemma 3.3.2 follows that there exists a subsequence $\{x^{(n_k)}(t;z_{n_k})\}_0^\infty$ of the sequence $\{x^{(n)}(t;z_n)\}_0^\infty$ that is uniformly convergent on the interval $[-h,T]$ and the limit $x(t)$ satisfies the relations

$$x(0) > x(T) \quad \text{and} \quad x(0) = w(0) - \delta^*, \tag{3.87}$$

$$x(t) > x^*(t) \quad \text{for} \;\; t \in [-h,T]. \tag{3.88}$$

Therefore, inequalities

$$x(T) < x(0) = w(0) - \delta^* = x^*(0) < x^*(T) \tag{3.89}$$

hold.

Inequality (3.89) contradicts inequality (3.88).

The obtained contradiction proves that the linear scalar impulsive differential-difference equation (3.78), (3.79) with initial condition $x(t;x_0) = x_0$ for $t \in [-h,0]$ has a solution $x(t;x_0)$ such that $x(0;x_0) = x(T;x_0)$.

We introduce the notation $g(t) = v(t) - x(t;x_0)$. Function $g(t)$ satisfies the inequalities (3.66)-(3.69) where $m(t) = g(t)$. According to Lemma 3.3.1 function $g(t)$ is nonpositive for $t \in [-h,T]$, i.e.

$$v(t) \leq x(t;x_0).$$

Similarly we can prove that $x(t;x_0) \leq w(t)$, i.e. $x(t;x_0) \in S(v,w)$. □

We will give an algorithm for constructing successive approximations and we will prove the application of the monotone-iterative technique for the periodic boundary value problem for a system of nonlinear impulsive differential-difference equations.

Theorem 3.3.1. *Let the following conditions be fulfilled:*

1. *The pair of functions $v,w \in PC([0,T],\mathbf{R}^n)$, where $v = (v_1,v_2,\dots,v_n)$ and $w = (w_1,w_2,\dots,w_n)$, is a pair of lower and upper quasisolutions of the periodic boundary value problem* (3.54), (3.55), (3.57), *and $v(t) \leq w(t)$ for $t \in [-h,T]$.*

2. *Function $f : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, where $f = (f_1, f_2,\dots, f_n)$ and $f_j(t,x,y) = f_j(t,x_j,[x]_{p_j},[x]_{q_j},y_j,[y]_{p_j},[y]_{q_j})$, is nondecreasing in $[x]_{p_j}$ and $[y]_{p_j}$, nonincreasing in $[x]_{q_j}$ and $[y]_{q_j}$ and for $x,y \in S(v,w)$, $y(t) \leq x(t)$, $t \in [0,T]$, $j = 1,2,\dots,n$ the inequality*

$$\begin{aligned} &f_j(t,x_j(t),[x(t)]_{p_j},[x(t)]_{q_j},x_j(t-h),[x(t-h)]_{p_j},[x(t-h)]_{q_j}) \\ &\quad - f_j(t,y_j(t),[y(t)]_{p_j},[y(t)]_{q_j},y_j(t-h),[y(t-h)]_{p_j},[y(t-h)]_{q_j}) \\ &\geq -M_j(x_j(t)-y_j(t)) - N_j((x_j(t-h)-y_j(t-h))), \end{aligned}$$

holds, where M_j,N_j, $j = 1,2,\dots,n$ are positive constants.

3. *Functions $I_i : \mathbf{R}^n \to \mathbf{R}^n$, where $I_i = (I_{i1},I_{i2},\dots,I_{in})$ and the functions $I_{ij}(x) = I_{ij}(x_j,[x]_{p_j},[x]_{q_j})$ are nondecreasing in $[x]_{p_j}$, nonincreasing in $[x]_{q_j}$ and for $x,y \in \Gamma_j(v,w)$, $y \leq x$ the inequalities*

$$I_{ij}(x_j,[x]_{p_j},[x]_{q_j}) - I_{ij}(y_j,[y]_{p_j},[y]_{q_j}) \geq -L_{ij}(x_j-y_j), \;\; j = 1,2,\dots,p,$$

hold, where L_{ij}, $i = 1,2,\dots,n$, $j = 1,2,\dots p$ are positive constants, $L_{ij} < 1$.

4. *The inequalities*

$$(M_j + N_j)p\tau < (1 - l_j)^p,$$
$$l_j = \max\{L_{ij} : \ i = 1, 2, \dots, p\} \tag{3.90}$$

hold, and the constant τ *is defined by the equality* (3.71).

Therefore, there exist two sequences of functions $\{v^{(k)}(t)\}_0^\infty$ *and* $\{w^{(k)}(t)\}_0^\infty$ *such that:*

a/ The sequences are increasing and decreasing respectively;

b/ The pair of functions $v^{(k)}(t)$, $w^{(k)}(t)$ *is a pair of lower and upper quasisolutions of the periodic boundary value problem for the system of nonlinear impulsive differential-difference equations* (3.54), (3.55), (3.57)*;*

c/ Both sequences are convergent on $[-h, T]$*;*

d/ The limits $V(t) = \lim_{k\to\infty} v^{(k)}(t)$, $W(t) = \lim_{k\to\infty} w^{(k}(t)$ *form a pair of minimal and maximal solutions of the periodic boundary value problem for the system of nonlinear impulsive differential-difference equations* (3.54), (3.55), (3.57)*;*

e/ If $u(t) \in S(v,w)$ *is a solution of the periodic boundary value problem for the system of nonlinear impulsive differential-difference equations* (3.54), (3.55), (3.57), *then* $V(t) \le u(t) \le W(t)$.

Proof. We fix two arbitrary functions $\eta, \mu \in S(v,w)$ and for all natural numbers $j : \ 1 \le j \le n$ we consider the periodic boundary value problem for the scalar linear impulsive differential-difference equation

$$u'(t) + M_j u(t) + N_j u(t-h)) = \psi_j(t, \eta, \mu) \ \text{ for } t \in [0, T] \ \ t \ne t_i, \tag{3.91}$$

$$u(t_i + 0) - u(t_i - 0) = -L_{ij} u(t_i) + \gamma_{ij}(\eta, \mu), \quad i = 1, 2, \dots, p, \tag{3.92}$$

$$u(t) = u(0), \quad t \in [-h, 0], \tag{3.93}$$

$$u(0) = u(T), \tag{3.94}$$

where $u \in \mathbf{R}$,

$$\begin{aligned} \psi_j(t, \eta, \mu) &= f_j(t, \eta_j(t), [\eta(t)]_{p_j}, [\mu(t)]_{q_j}, \eta_j(t-h), [\eta(t-h)]_{p_j}, [\mu(t-h)]_{q_j}) \\ &\quad + M_j \eta_j(t) + N_j \eta_j(t-h), j = 1, 2, \dots, n, \\ \gamma_{ij}(\eta, \mu) &= I_{ij}(\eta_j(t_i), [\eta(t_i)]_{p_j}, [\mu(t_i)]_{q_j}) + L_{ij} \eta_j(t_i). \end{aligned}$$

According to Lemma 3.3.2 and Lemma 3.3.1 the periodic boundary value problem (3.91)–(3.94) has unique solution for the fixed pair of functions $\eta, \mu \in S(v,w)$.

For every two functions $\eta, \mu \in S(v,w)$ such that $\eta(t) \le \mu(t)$ for $t \in [-h, T]$, we define the operator $\Omega : S(v,w) \times S(v,w) \to S(v,w)$ by the equality $\Omega(\eta, \mu) = x(t)$, where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ and $x_j(t)$ is the unique solution of the periodic boundary value problem for scalar impulsive equation (3.91)–(3.94) for the pair of functions η, μ . According to Lemma 3.2.2 the inequalities $v \le \Omega(v,w)$ and $w \ge \Omega(w,v)$ hold.

Let $\eta, \mu \in S(v,w)$ be arbitrary functions such that $\eta(t) \le \mu(t)$ for $t \in [-h, T]$. Inroduce the notations $x^{(1)} = \Omega(\eta, \mu), x^{(2)} = \Omega(\mu, \eta)$, and $g = x^{(1)} - x^{(2)}$, where $g = (g_1, g_2, \dots, g_n)$. According to Lemma 3.3.1 the functions $g_j(t)$ are nonpositive, i.e. $\Omega(\eta, \mu) \le \Omega(\mu, \eta)$.

Define the sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)} \equiv v, \qquad w^{(0)} \equiv w,$$
$$v^{(k+1)} = \Omega(v^{(k)}, w^{(k)}), \qquad w^{(k+1)} = \Omega(w^{(k)}, v^{(k)}),\ k \geq 0.$$

Functions $v^{(k)}(t)$ and $w^{(k)}(t)$, $(k = 0, 1, 2, \dots)$ for $t \in [-h, T]$ satisfy the inequalities

$$v^{(0)}(t) \leq v^{(1)}(t) \leq \cdots \leq v^{(k)}(t) \leq w^{(k)}(t) \leq \cdots \leq w^{(1)}(t) \leq w^{(0)}(t). \tag{3.95}$$

Both sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are convergent on the interval $[-h, T]$. We will prove that their limits $V(t)$ and $W(t)$ form a pair of minimal and maximal quasisolutions of the periodic boundary value problem (3.54), (3.55), (3.57). From the definition of functions $v^{(k)}(t)$ and $w^{(k)}(t)$, where $v^{(k)} = (v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)})$, and $w^{(k)} = (w_1^{(k)}, w_2^{(k)}, \dots, w_n^{(k)})$, follows that these functions satisfy the periodic boundary value problems ($j = 1, 2, \dots, n$)

$$(v_j^{(k)}(t))' + M_j v_j^{(k)}(t) + N_j v_j^{(k)}(t-h)) = \psi_j(t, v^{(k-1)}, w^{(k-1)}), \tag{3.96}$$

$$(w_j^{(k)}(t))' + M_j w_j^{(k)}(t) + N_j w_j^{(k)}(t-h)) = \psi_j(t, w^{(k-1)}, v^{(k-1)})$$
$$\text{for } t \in [0, T]\ \ t \neq t_i,$$

$$v_j^{(k)}(t_i+0) - v_j^{(k)}(t_i-0) = -L_{ij} v_j^{(k)}(t_i) + \gamma_{ij}(v^{(k-1)}, w^{(k-1)}), \tag{3.97}$$

$$w_j^{(k)}(t_i+0) - w_j^{(k)}(t_i-0) = -L_{ij} w_j^{(k)}(t_i) + \gamma_{ij}(w^{(k-1)}, v^{(k-1)}),$$
$$i = 1, 2, \dots, p,$$

$$v_j^{(k)}(t) = v_j^{(k)}(0), \quad w_j^{(k)}(t) = w_j^{(k)}(0), \quad t \in [-h, 0], \tag{3.98}$$

$$v_j^{(k)}(0) = v_j^{(k)}(T), \qquad w_j^{(k)}(0) = w_j^{(k)}(T). \tag{3.99}$$

Consider equalities (3.96)–(3.99) as $k \to \infty$ and denote $V_j(t) = \lim_{k\to\infty} v_j^{(k)}(t)$, $W_j(t) = \lim_{k\to\infty} w_j^{(k)}(t)$. We define the functions $V(t)$ and $W(t)$, where $V = (V_1, V_2, \dots, V_n)$ and $W = (W_1, W_2, \dots, W_n)$. The pair of functions $V(t)$ and $W(t)$ is a pair of quasisolutions of the periodic boundary value problem (3.54), (3.55), (3.57). Let $u, z \in S(v, w)$ be a pair of quasisolutions of the periodic boundary value problem (3.54), (3.55), (3.57). From inequalities (3.95) follows that there exists a natural number k such that $v^{(k)}(t) \leq u(t) \leq w^{(k)}(t)$ and $v^{(k)}(t) \leq z(t) \leq w^{(k)}(t)$ for $t \in [-h, T]$. We introduce the notation $g(t) = v^{(k+1)}(t) - u(t)$, $g = (g_1, g_2, \dots, g_n)$. According to Lemma 3.2.1 the inequalities $g_j(t) \leq 0$ hold for $t \in [-h, T]$, i.e. $v^{(k+1)}(t) \leq u(t)$.

Similarly to the above proofs, we prove the inequalities $w^{(k+1)}(t) \geq u(t)$ and $v^{(k+1)}(t) \leq z(t) \leq w^{(k+1)}(t)$ for $t \in [-h, T]$. Therefore, the pair of functions $V(t)$ and $W(t)$ is a pair of minimal and maximal solutions of the periodic boundary value problem (3.54), (3.55), (3.57).

Let $u(t) \in S(v, w)$ be a solution of the periodic boundary value problem (3.54), (3.55), (3.57). Consider the pair of functions (u, u) that is a pair of quasisolutions of the periodic boundary value problem (3.54), (3.55), (3.57). Therefore, the inequalities $V(t) \leq u(t) \leq W(t)$ hold for $t \in [-h, T]$. □

As a partial case of the obtained in this section results we obtain results for the periodic boundary value problem for different types of equations such as:

– systems of impulsive differential equations with fixed moments of impulses ([13], [72]);

– scalar impulsive differential-difference equations with fixed moments of impulses ([93]);

– differential-difference equations without impulses (scalar and n-dimensional case) ([88] and cited therein bibliography).

Chapter 4

Method of Quasilinearization for Impulsive Differential Equations

The method of quasilinearization is a practically useful method for obtaining approximate solutions of nonlinear problems. The origin of this method lies in the theory of dynamic programming [20]. The quasilinearization method is a Taylor series numerical method in which the truncation is chosen so that the convergence of the iterates is quadratic. Many authors have applied this method to finding approximate solutions of different types of first and second order ordinary differential equations ([34], [91], [101], [112], [118]). For scalar impulsive differential equations some results are obtained in [42], [43]. In this chapter the quasilinearization method have been employed on different problems for impulsive differential equations.

Part of the results presented in this chapter are published in [6], [7], [8], [10], [52], [53], [76], [78].

Let points $t_k \in (0,T)$ be fixed such that $t_{k+1} > t_k$, $k = 1,2,\dots,p$, $t_0 = 0$, $t_{p+1} = T$.

4.1. Method of Quasilinearization for the Initial Value Problem for Systems of Impulsive Differential Equations

Consider the initial value problem for the system of nonlinear impulsive differential equations

$$x' = F(t,x(t)) \quad \text{for} \quad t \in [0,T], \;\; t \neq t_k, \tag{4.1}$$

$$x(t_k+0) = G_k(x(t_k)), \; k = 1,2,\dots,p, \tag{4.2}$$

$$x(0) = x_0, \tag{4.3}$$

where $x \in \mathbf{R}^n$, $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$, $G_k : \mathbf{R}^n \to \mathbf{R}^n, (k = 1,2,\dots,p)$. We will assume that $F(t,x) = f(t,x) + g(t,x)$ and $G_k(x) = I_k(x) + J_k(x), k = 1,2,\dots,p$, where the properties of the functions f, g, I_k and J_k will be given later.

Let $x = (x_1,x_2,\dots,x_n)$. Then $||x|| = max\{|x_1|,|x_2|,\dots,|x_n|\}$.

Definition 26. Functions $\alpha(t),\beta(t)\in PC^1([0,T],\mathbf{R}^n)$ are called *a mixed couple of lower and upper solutions* of the initial value problem for the system of nonlinear impulsive differential equations (4.1), (4.2), (4.3) if the following inequalities are satisfied:

$$\alpha'(t)\le f(t,\alpha(t))+g(t,\beta(t)),$$
$$\beta'(t)\ge f(t,\beta(t))+g(t,\alpha(t)) \quad \text{for} \quad t\in[0,T],t\neq t_k, \tag{4.4}$$

$$\alpha(t_k+0)\le I_k(\alpha(t_k))+J_k(\beta(t_k)),$$
$$\beta(t_k+0)\ge I_k(\beta(t_k))+J_k(\alpha(t_k)), \tag{4.5}$$
$$\alpha(0)\le x_0\le\beta(0). \tag{4.6}$$

Definition 27. Functions $\alpha(t),\beta(t)\in PC^1([0,T],\mathbf{R}^n)$ are called *a mixed couple of solutions* of the initial value problem for the system of nonlinear impulsive differential equations (4.1), (4.2), (4.3) if the relations (4.4) - (4.6) are fulfilled only for equalities.

Let functions $\alpha,\beta\in PC([0,T]),\mathbf{R}^n)$ be such that $\alpha(t)\le\beta(t)$.
Consider the sets:

$$S(\alpha,\beta)=\{u\in PC([0,T],\mathbf{R}^N):\ \alpha(t)\le u(t)\le\beta(t),t\in[0,T]\}, \tag{4.7}$$
$$\Omega(\alpha,\beta)=\{(t,x)\in[0,T]\times\mathbf{R}^N:\ \ \alpha(t)\le x\le\beta(t)\}, \tag{4.8}$$
$$\Gamma_i(\alpha,\beta)=\{x\in\mathbf{R}^N,\alpha(t_i)\le x\le\beta(t_i)\},\ \ i=1,2\ldots,p. \tag{4.9}$$

We will say that function $F:[0,T]\times\mathbf{R}^N\to\mathbf{R}^N,\ \ F=(F_1,F_2,\ldots,F_N)$ satisfies the conditions (H) if there exists a natural number $m:1\le m\le N$ such that:

$(H1)$ Functions $F_i(t,x),i=1,2,\ldots,m$ are nondecreasing in x_l and nonincreasing in x_j where $l=1,2,\ldots,m,l\neq i,j=m+1,m+2,\ldots,N.$

$(H2)$ Functions $F_j(t,x),j=m+1,m+2,\ldots,N$ are nonincreasing in x_i and nondecreasing in x_l where $i=1,2,\ldots,m,\ \ l=m+1,m+2,\ldots,N,l\neq j.$

We will use the following result for ordinary differential inequalities:

Lemma 4.1.1. *Assume that*

1. *Function* $F\in C([0,T]\times\mathbf{R}^N,\mathbf{R}^N),F=(F_1,F_2,\ldots,F_N)$ *and satisfies the conditions* (H).

2. *Function* $u\in C^1([0,T],\mathbf{R}^N)$ *and*

$$u_i'(t)\le F_i(t,u(t)),\ \ u_j'(t)\ge F_j(t,u(t)),\ \ u_i(0)\le 0\le u_j(0)$$

where $i=1,2,\ldots,m,\ \ j=m+1,m+2,\ldots N.$

Then $u_i(t)\le 0$ *and* $u_j(t)\ge 0,\ \ i=1,2\ldots,m,j=m+1,m+2,\ldots N$ *for* $t\in[0,T].$

In the proof of the main results we will use the following result for impulsive differential inequalities.

Lemma 4.1.2. *Let the following conditions be fulfilled:*

1. *Function* $F \in PC([0,T] \times \mathbf{R}^N, \mathbf{R}^N)$ *and conditions* (H) *are satisfied.*

2. *Functions* $I_k : \mathbf{R}^N \to \mathbf{R}^N, I_k = (I_{k1}, I_{k2}, \dots, I_{kN}), k = 1,2,\dots,r$ *and for* $x_i \leq 0, \ \ x_j \geq 0, \ \ i = 1,2,\dots,m, \ \ j = m+1, m+2, \dots N$ *the inequalities* $I_{ki}(x) \leq 0$ *and* $I_{kj}(x) \geq 0$ *hold.*

3. *Function* $u \in PC^1([0,T], \mathbf{R}^N)$ *and*

$$u_i'(t) \leq F_i(t, u(t)), \ \ u_j'(t) \geq F_j(t, u(t)) \quad for \ \ t \in [0,T], t \neq t_k \tag{4.10}$$

$$u_i(t_k + 0) \leq I_{ki}(u(t_k)), \ \ u_j(t_k + 0) \geq I_{kj}(u(t_k)), \ \ k = 1,2,\dots,p, \tag{4.11}$$

$$u_i(0) \leq 0, \ \ u_j(0) \geq 0. \tag{4.12}$$

Then $u_i(t) \leq 0$*, and* $u_j(t) \geq 0, \ \ i = 1,2,\dots,m, \ \ j = m+1, m+2, \dots N$ *for* $t \in [0,T]$.

Proof. Let $t \in [0, t_1]$. According to Lemma 4.1.1 from condition 1 and inequalities (4.10), (4.12) we conclude that $u_i(t) \leq 0$ and $u_j(t) \geq 0$ for $i = 1,2,\dots,m, \ j = m+1, m+2, \dots N$ and for $t \in [0, t_1]$. From the inequality (4.11) and the condition 2 of Lemma 4.1.2 we obtain that

$$u_i(t_1 + 0) \leq I_{1i}(u(t_1)) \leq 0, \quad \text{and} \quad u_j(t_1 + 0)) \geq I_{1j}(u(t_1)) \geq 0. \tag{4.13}$$

From Lemma 4.1.1 and inequalities (4.10), (4.13) we obtain the inequalities $u_i(t) \leq 0$ and $u_j(t) \geq 0$ for $i = 1,2,\dots,m, \ \ j = m+1, m+2, \dots N$ and for $t \in (t_1, t_2]$. By the help with the method of mathematical induction we obtain that the conclusion of Lemma 4.1.2 is true. □

In the case when $m = N$ as a corollary of Lemma 4.1.2 we obtain the following result.

Lemma 4.1.3. *Let the following conditions be fulfilled:*

1. *Function* $F \in PC([0,T] \times \mathbf{R}^N, \mathbf{R}^N)$ *and it is quasimonotone nondecreasing in x.*

2. *Functions* $I_k : \mathbf{R}^N \to \mathbf{R}^N, k = 1,2,\dots,p$ *and* $I_k(x) \leq 0$ *for* $x \leq 0$.

3. *Function* $u \in PC^1([0,T], \mathbf{R}^N)$ *satisfies the inequalities*

$$\begin{aligned}
&u'(t) \leq F(t,u) \quad for \quad t \in [0,T], t \neq t_k, \\
&u(t_k + 0) \leq I_k(u(t_k)), \\
&u(0) \leq 0.
\end{aligned}$$

Then $u(t) \leq 0$ *for* $t \in [0,T]$.

Furthermore, we will use the following form of the Mean Value Theorem.

Lemma 4.1.4 ([92]). *Let function* $F \in C^1(D, \mathbf{R}^n)$*, where* $D \subset \mathbf{R}^n$ *is a convex set. Then*

$$F(x) - F(y) = \Big(\int_0^1 F_x'(\lambda x + (1-\lambda) y) d\lambda \Big)(x - y).$$

Lemma 4.1.5 ([17]). *Let* $u \in PC([t_0, \infty), \mathbf{R}^n)$ *and*

$$\begin{aligned}
&u'(t) \leq Au(t) + f(t) \quad for \ \ t > t_0, \ \ t \neq t_k, \\
&u(t_k + 0) \leq B_k u(t_k) + f_k,
\end{aligned}$$

where $f \in PC([t_0,\infty),\mathbf{R}^n)$, $A = (a_{ij}) \in \mathbf{R}^{n\times n}$, $B_k = (b^k_{ij}) \in \mathbf{R}^{n\times n}$, $a_{ij} \geq 0, b^k_{ij} \geq 0$, $1 \leq i \leq n, 1 \leq j \leq n$, $f_k \in \mathbf{R}^n$, $f_k \geq 0, k = 1,2,\dots,p$.

Then for $t > t_0$ *we obtain*

$$u(t) \leq W(t,t_0)u(t_0) + \int_{t_0}^{t} W(t,s)f(s)ds + \sum_{t_0<t_k<t} W(t,t_k+0)f_k,$$

where $W(t,s) = e^{A(t-s)}\prod_{j:s\leq t_j<t}(E+B_j)$, *E is the* $(n\times n)$*-identity matrix.*

We apply the method of quasilinearization for approximate solving of the initial value problem (4.1), (4.2), (4.3) in the case when there exists a mixed couple of lower and upper solutions. One of the main condition is the monotonicity of the right parts. Two different cases are considered.

Theorem 4.1.1. *Let the following conditions be fulfilled:*

1. *Functions* $\alpha_0, \beta_0 \in PC^1([0,T],\mathbf{R}^n)$ *are a mixed couple of upper and lower solutions of the initial value problem* (4.1), (4.2), (4.3), *and* $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$.

2. *Functions* $f,g \in C([0,T]\times\mathbf{R}^n,\mathbf{R}^n)$ *and the derivatives* $f_x(t,x), g_x(t,x)$ *exist and they are continuous on* $\Omega(\alpha_0,\beta_0)$, $f_x(t,x)$ *is nondecreasing in x ,* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$ *and*

$$f_x(t,x) - f_x(t,y) \leq L^{(1)}||x-y||, \quad g_x(t,x) - g_x(t,y) \leq L^{(2)}||x-y||$$

where $L^{(1)}, L^{(2)}$ *are constant matrices with positive elements.*

3. *The derivatives* $I'_k(x), J'_k(x), (k = 1,2,\dots,p)$ *exist and they are continuous on* $\Gamma_k(\alpha_0,\beta_0)$, *the derivatives* $I'_k(x)$ *are nondecreasing,* $J'_k(x)$ *are nonincreasing,* $I'_k(x) \geq 0 \geq J'_k(x)$, *and*

$$I'_k(x) - I'_k(y) \leq M^{(1)}_k||x-y||, \quad J'_k(x) - J'_k(y) \leq M^{(2)}_k||x-y||$$

for $x,y \in \Gamma_k(\alpha_0,\beta_0)$, *where* $M^{(1)}_k, M^{(2)}_k$ *are constant matrices with positive elements.*

4. *For each* $\eta \in S(\alpha_0,\beta_0)$ *the function* $f_x(t,\eta)x$ *is nondecreasing in x and* $g_x(t,\eta)x$ *is nonincreasing in x.*

Then there exist two sequences $\{\alpha_n(t)\}_0^\infty$ *and* $\{\beta_n(t)\}_0^\infty$ *such that*

a/ The sequences are monotone increasing and monotone decreasing, respectively;

b/ Functions $\alpha_n(t), \beta_n(t)$ *form a mixed couple of upper and lower solutions of the initial value problem* (4.1), (4.2), (4.3)*;*

c/ Both sequences converge uniformly to the unique solution of the initial value problem (4.1), (4.2), (4.3) *in* $S(\alpha_0,\beta_0)$ *for* $t \in (t_k,t_{k+1}]$, $k = 0,1,2,\dots,p$, $t_0 = 0$, $t_{p+1} = T$*;*

d/ The convergence is quadratic, i.e. there exist constants $\lambda_i \geq 0, i = 1,2,3,4$ *such that*

$$|||x(t) - \alpha_{n+1}(t)||| \leq \lambda_1|||x(t) - \alpha_n(t)|||^2 + \lambda_2|||\beta_n(t) - x(t)|||^2$$

and

$$|||\beta_{n+1}(t) - x(t)||| \leq \lambda_3|||x(t) - \alpha_n(t)|||^2 + \lambda_4|||\beta_n(t) - x(t)|||^2,$$

where $|||u||| = sup\{||u(t)|| : t \in [0,T]\}$.

Proof. Consider the initial value problem for the following system of linear impulsive differential equations

$$
\begin{aligned}
x' &= f(t,\alpha_0) + f_x(t,\alpha_0)(x-\alpha_0) + g(t,\beta_0) + g_x(t,\alpha_0)(y-\beta_0),\\
y' &= f(t,\beta_0) + f_x(t,\alpha_0)(y-\beta_0) + g(t,\alpha_0) + g_x(t,\alpha_0)(x-\alpha_0),\\
&\qquad \text{for } t\in[0,T], t\neq t_k,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
x(t_k+0) &= I_k(\alpha_0(t_k)) + I_k'(\alpha_0(t_k))(x(t_k)-\alpha_0(t_k)) + J_k(\beta_0(t_k))\\
&\quad + J_k'(\alpha_0(t_k))(y(t_k)-\beta_0(t_k)),\\
y(t_k+0) &= I_k(\beta_0(t_k)) + I_k'(\alpha_0(t_k))(y(t_k)-\beta_0(t_k)) + J_k(\alpha_0(t_k))\\
&\quad + J_k'(\alpha_0(t_k))(x(t_k)-\alpha_0(t_k)),\ \ k=1,2,\dots,p,
\end{aligned}
\tag{4.15}
$$

$$x(0) = x_0 = y(0). \tag{4.16}$$

The initial value problem for the system of linear impulsive differential equations (4.14), (4.15), (4.16) has a unique solution $\alpha_1(t), \beta_1(t)$ for $t\in[0,T]$.

We will prove that $\alpha_1, \beta_1 \in S(\alpha_0,\beta_0)$. Set $p=\alpha_0-\alpha_1$ and $q=\beta_0-\beta_1$. Then according to the choice of the functions $\alpha_1(t)$ and $\beta_1(t)$ we obtain that

$$
\begin{aligned}
&p'(t) = \alpha_0' - \alpha_1' \le f(t,\alpha_0) + g(t,\beta_0)\\
&\quad -[f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g(t,\beta_0) + g_x(t,\alpha_0)(\beta_1-\beta_0)]\\
&\quad = f_x(t,\alpha_0)p + g_x(t,\alpha_0)q \quad \text{for} \quad t\in[0,T], t\neq t_k,\\
&q'(t) \ge f_x(t,\alpha_0)q + g_x(t,\alpha_0)p,\\
&p(t_k+0) = \alpha_0(t_k+0) - \alpha_1(t_k+0) \le I_k(\alpha_0(t_k)) + J_k(\beta_0(t_k))\\
&\quad -[I_k(\alpha_0(t_k)) - I_k'(\alpha_0(t_k))p(t_k) + J_k(\beta_0(t_k)) - J_k'(\beta_0(t_k))q(t_k)]\\
&\quad = I_k'(\alpha_0(t_k))p(t_k) + J_k'(\alpha_0(t_k))q(t_k),\\
&q(t_k+0) \ge I_k'(\alpha_0(t_k))q(t_k) + J_k'(\alpha_0(t_k))p(t_k),\\
&p(0) \le 0 \le q(0).
\end{aligned}
$$

According to Lemma 4.1.2 for $N=2n$ and $m=n$ we have $p(t)\le 0$ and $q(t)\ge 0$ for $t\in[0,T]$ which implies that $\alpha_0(t)\le\alpha_1(t)$ and $\beta_0(t)\ge\beta_1(t)$ for $t\in[0,T]$.

Consider function $p(t)=\alpha_1(t)-\beta_1(t), t\in[0,T]$. From (4.16) follows that $p(0)=0$. According to condition 2 of Theorem 4.1.1 and Lemma 4.1.4 we obtain for $t\in[0,T], t\neq t_k$, that

$$
\begin{aligned}
p' &= f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g(t,\beta_0) + g_x(t,\alpha_0)(\beta_1-\beta_0)\\
&\quad -[f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g(t,\alpha_0) + g_x(t,\alpha_0)(\alpha_1-\alpha_0)]\\
&= \Big(\int_0^1 f_x(t,\alpha_0\lambda + (1-\lambda)\beta_0)d\lambda\Big)(\alpha_0-\beta_0)\\
&\quad + \Big(\int_0^1 g_x(t,\beta_0\lambda + (1-\lambda)\alpha_0)d\lambda\Big)(\beta_0-\alpha_0)\\
&\quad + f_x(t,\alpha_0)(\alpha_1-\alpha_0-\beta_1+\beta_0) + g_x(t,\alpha_0)(\beta_1-\beta_0-\alpha_1+\alpha_0)
\end{aligned}
$$

$$\leq f_x(t,\alpha_0)(\alpha_0-\beta_0)+g_x(t,\alpha_0)(\beta_0-\alpha_0)+f_x(t,\alpha_0)(p-\alpha_0+\beta_0)$$
$$+g_x(t,\alpha_0)(\alpha_0-\beta_0-p)$$
$$=[f_x(t,\alpha_0)-g_x(t,\alpha_0)]p. \tag{4.17}$$

According to condition 3 of Theorem 4.1.1 and Lemma 4.1.4 we obtain

$$\begin{aligned}p(t_k+0) &= I_k(\alpha_0(t_k))-I_k(\beta_0(t_k))+J_k(\beta_0(t_k))-J_k(\alpha_0(t_k))\\ &\quad +I'_k(\alpha_0(t_k))[p(t_k)-\alpha_0(t_k)+\beta_0(t_k)]\\ &\quad +J'_k(\alpha_0(t_k)[\alpha_0(t_k)-\beta_0(t_k)-p(t_k)]\end{aligned}$$
$$\leq [I'_k(\alpha_0(t_k))-J'_k(\alpha_0(t_k))]p(t_k),\ k=1,2,\ldots,p. \tag{4.18}$$

According to Lemma 4.1.3 from inequalities (4.17), (4.18) follows that function $p(t)$ is nonpositive in $[0,T]$, i.e. $\alpha_1(t)\leq\beta_1(t)$. Therefore $\alpha_1,\beta_1\in S(\alpha_0,\beta_0)$.

We will prove that the couple of functions $\alpha_1(t),\beta_1(t)$ is a mixed couple of lower and upper solutions of the initial value problem (4.1), (4.2), (4.3). Indeed, from Lemma 4.1.4 and the monotonicity of the derivatives of functions $f(t,x),g(t,x)$ we obtain for $t\in[0,T],t\neq t_k$

$$\begin{aligned}\alpha'_1 &= f(t,\alpha_1)+g(t,\beta_1)+[f(t,\alpha_0)-f(t,\alpha_1)]\\ &\quad +[g(t,\beta_0)-g(t,\beta_1)]+f_x(t,\alpha_0)(\alpha_1-\alpha_0)+g_x(t,\alpha_0)(\beta_1-\beta_0)\\ &= f(t,\alpha_1)+g(t,\beta_1)+\Big(\int_0^1 f_x(t,\lambda\alpha_0+(1-\lambda)\alpha_1)d\lambda\Big)(\alpha_0-\alpha_1)\\ &\quad +\Big(\int_0^1 g_x(t,\lambda\beta_0+(1-\lambda)\beta_1)d\lambda\Big)(\beta_0-\beta_1)\\ &\quad +f_x(t,\alpha_0)(\alpha_1-\alpha_0)+g_x(t,\alpha_0)(\beta_1-\beta_0)\\ &\leq f(t,\alpha_1)+g(t,\beta_1)+[g_x(t,\beta_1)-g_x(t,\alpha_0)](\beta_0-\beta_1)\\ &\leq f(t,\alpha_1)+g(t,\beta_1),\end{aligned}$$

$$\begin{aligned}\beta'_1 &= f(t,\beta_1)+g(t,\alpha_1)+[f(t,\beta_0)-f(t,\beta_1)]\\ &\quad +[g(t,\alpha_0)-g(t,\alpha_1)]+f_x(t,\alpha_0)(\beta_1-\beta_0)+g_x(t,\alpha_0)(\alpha_1-\alpha_0)\end{aligned}$$
$$\geq f(t,\beta_1)+g(t,\alpha_1). \tag{4.19}$$

Analogously we can prove that

$$\begin{aligned}\alpha_1(t_k+0) &= I_k(\alpha_1(t_k))+[I_k(\alpha_0(t_k))-I_k(\alpha_1(t_k))]\\ &\quad +J_k(\beta_1(t_k))+[J_k(\beta_0(t_k))-J_k(\beta_1(t_k))]\\ &\quad +I'_k(\alpha_0(t_k))[\alpha_1(t_k)-\alpha_0(t_k)]+J'_k(\alpha_0(t_k)[\beta_1(t_k)-\beta_0(t_k)]\\ &\leq I_k(\alpha_1(t_k))+J_k(\beta_1(t_k)),\end{aligned}$$
$$\beta_1(t_k+0)\geq I_k(\beta_1(t_k))+J_k(\alpha_1(t_k)). \tag{4.20}$$

From inequalities (4.19), (4.20) and equality (4.16) we conclude that functions $\alpha_1(t),\beta_1(t)$ form a mixed couple of lower and upper solutions of the initial value problem (4.1), (4.2), (4.3).

Consider the initial value problem (4.14), (4.15), (4.16), where functions $\alpha_0(t)$ and $\beta_0(t)$ are replaced by functions $\alpha_1(t)$ and $\beta_1(t)$. This initial value problem has a unique solution $\alpha_2(t), \beta_2(t)$. Analogously to the above proofs, we can prove that $\alpha_2(t) \le \beta_2(t)$, $\alpha_2, \beta_2 \in S(\alpha_1, \beta_1)$.

Following the above procedure, we can construct functions $\alpha_{n+1}(t)$, $\beta_{n+1}(t)$, $n = 1, 2, \ldots$ such that they are solutions of the initial value problem for the system of linear impulsive differential equations

$$\begin{aligned} x' &= f(t,\alpha_n) + f_x(t,\alpha_n)(x-\alpha_n) + g(t,\beta_n) + g_x(t,\alpha_n)(y-\beta_n), \\ y' &= f(t,\beta_n) + f_x(t,\alpha_n)(y-\beta_n) + g(t,\alpha_n) + g_x(t,\alpha_n)(x-\alpha_n), \end{aligned}$$

$$\text{for } t \in [0,T], t \neq t_k, \tag{4.21}$$

$$\begin{aligned} x(t_k+0) &= I_k(\alpha_n(t_k)) + I_k'(\alpha_n(t_k))(x(t_k)-\alpha_n(t_k)) + J_k(\beta_n(t_k)) \\ &\quad + J_k'(\alpha_n(t_k))(y(t_k)-\beta_n(t_k)), \end{aligned} \tag{4.22}$$

$$\begin{aligned} y(t_k+0) &= I_k(\beta_n(t_k)) + I_k'(\alpha_n(t_k))(y(t_k)-\beta_k(t_k)) + J_k(\alpha_n(t_k)) \\ &\quad + J_k'(\alpha_n(t_k))(x(t_k)-\alpha_n(t_k)), \end{aligned}$$

$$x(0) = x_0 = y(0). \tag{4.23}$$

Assume that $\alpha_n, \beta_n \in S(\alpha_{n-1}, \beta_{n-1})$, and $\alpha_n(t) \le \beta_n(t)$. Define the functions $p(t) = \alpha_n(t) - \alpha_{n+1}(t)$ and $q(t) = \beta_n(t) - \beta_{n+1}(t)$ for $t \in [0,T]$. From Lemma 4.1.4 and the monotonicity of the derivatives of the functions $f(t,x), g(t,x), I_k(x), J_k(x)$ we obtain that

$$\begin{aligned} p'(t) &= f(t,\alpha_{n-1}) - f(t,\alpha_n) + g(t,\beta_{n-1}) - g(t,\beta_n) \\ &\quad + f_x(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) - f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &\quad + g_x(t,\alpha_{n-1})(\beta_n - \beta_{n-1}) - g_x(t,\alpha_n)(\beta_{n+1} - \beta_n) \\ &\le f_x(t,\alpha_{n-1})(\alpha_{n-1} - \alpha_n) + g_x(t,\beta_{n-1})(\beta_{n-1} - \beta_n) \\ &\quad + f_x(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + g_x(t,\alpha_{n-1})(\beta_n - \beta_{n-1}) \\ &\quad + f_x(t,\alpha_n)p + g_x(t,\alpha_n)q \\ &\le f_x(t,\alpha_n)p + g_x(t,\alpha_n)q, \end{aligned}$$

$$q'(t) \ge g_x(t,\alpha_n)p + f_x(t,\alpha_n)q \quad \text{for } t \neq t_k, t \in [0,T], \tag{4.24}$$

$$p(t_k+0) \le I_k'(\alpha_n(t_k))p(t_k) + J_k'(\alpha_n(t_k))q(t_k),$$

$$q(t_k+0) \ge J_k'(\alpha_n(t_k))p(t_k) + I_k'(\alpha_n(t_k))q(t_k), \tag{4.25}$$

$$p(0) = q(0) = 0. \tag{4.26}$$

The relations (4.24), (4.25), (4.26) and Lemma 4.1.2 yield that $p(t) \le 0$ and $q(t) \ge 0$ on $[0,T]$, i.e. $\alpha_n(t) \le \alpha_{n+1}(t)$ and $\beta_n(t) \ge \beta_{n+1}(t)$. We set $p(t) = \alpha_{n+1}(t) - \beta_{n+1}(t)$ for $t \in [0,T]$ and we get

$$p'(t) = f(t,\alpha_n) - f(t,\beta_n) + f_x(t,\alpha_n)(p - \alpha_n + \beta_n)$$

$$
\begin{aligned}
& +g(t,\beta_n)-g(t,\alpha_n)+g_x(t,\alpha_n)(\alpha_n-\beta_n-p) \\
\leq\ & f_x(t,\alpha_n)(\alpha_n-\beta_n)-g_x(t,\alpha_n)p \\
& +f_x(t,\alpha_n)p-f_x(t,\alpha_n)(\alpha_n-\beta_n) \\
& +g_x(t,\alpha_n)(\beta_n-\alpha_n)+g_x(t,\alpha_n)(\alpha_n-\beta_n) \\
=\ & [f_x(t,\alpha_n)-g_x(t,\alpha_n)]p, \\
p(t_k+0) \leq\ & [I_k'(\alpha_n(t_k))-J_k'(\alpha_n(t_k))]p(t_k), \\
p(0) =\ & 0.
\end{aligned}
$$

Using Lemma 4.1.3, we find that $p(0) \leq 0$ on $[0,T]$. Therefore, $\alpha_{n+1}(t) \leq \beta_{n+1}(t)$ and $\alpha_{n+1}, \beta_{n+1} \in S(\alpha_n, \beta_n)$.

From the properties of the derivatives of functions $f(t,x), g(t,x)$ and Lemma 4.1.4 we obtain that for $t \in [0,T], t \neq t_k$ the following inequalities

$$
\begin{aligned}
\alpha_{n+1}' =\ & f(t,\alpha_{n+1})+g(t,\beta_{n+1})+[f(t,\alpha_n)-f(t,\alpha_{n+1})] \\
& +[g(t,\beta_n)-g(t,\beta_{n+1})]+f_x(t,\alpha_n)(\alpha_{n+1}-\alpha_n) \\
& +g_x(t,\alpha_n)(\beta_{n+1}-\beta_n) \\
& \leq f(t,\alpha_{n+1})+g(t,\beta_{n+1}),
\end{aligned} \tag{4.27}
$$

$$
\beta_{n+1}' \geq f(t,\beta_{n+1})+g(t,\alpha_{n+1})
$$

hold.

Analogously, we can prove that

$$
\alpha_{n+1}(t_k+0) \leq I_k(\alpha_{n+1}(t_k))+J_k(\beta_{n+1}(t_k)),
$$

$$
\beta_{n+1}(t_k+0) \geq I_k(\beta_{n+1}(t_k))+J_k(\alpha_{n+1}(t_k)). \tag{4.28}
$$

Inequalities (4.27) and (4.28) imply that the couple of functions $\alpha_{n+1}(t), \beta_{n+1}(t)$ form a mixed couple of lower and upper solutions of the initial value problem (4.1), (4.2), (4.3).

Therefore the inequalities

$$
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \tag{4.29}
$$

hold.

The sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ are uniformly bounded and equi - continuous on intervals $(t_k, t_{k+1}], k = 0,1,2,\ldots,p$. Both sequences are uniformly convergent. We denote

$$
\lim_{n\to\infty} \alpha_n(t) = u(t), \quad \lim_{n\to\infty} \beta_n(t) = v(t).
$$

From the uniform convergence and the definition of functions $\alpha_n(t)$ and $\beta_n(t)$ it follows the validity of the inequalities

$$
\alpha_0(t) \leq u(t) \leq v(t) \leq \beta_0(t). \tag{4.30}
$$

From the definition of function $u(t)$ and $v(t)$, and equalities (4.21), (4.22), and (4.23) follows that functions $u(t), v(t)$ form a mixed couple of solutions of the initial value problem

(4.1), (4.2), (4.3). Since functions f, g, I_k and J_k are Lipschitz in $\Omega(\alpha_0, \beta_0)$ and $\Gamma_k(\alpha_0, \beta_0)$ correspondingly, one can employ uniqueness of solutions of initial value problems to argue that if x is the unique solution of the initial value problem (4.1), (4.2), (4.3) in the set $S(\alpha_0, \beta_0)$, then $u = v = x$.

We will prove that the convergence is quadratic.

Define functions $p_n(t) = x(t) - \alpha_n(t)$ and $q_n(t) = \beta_n(t) - x(t),\ t \in [0,T], n = 0,1,2,\ldots$. According to the above proof $p_n(0) = q_n(0) = 0, p_n(t) \geq 0, q_n(t) \geq 0$ for $t \in [0,T], n = 0,1,2,\ldots$.

From the properties of functions $f(t,x), g(t,x)$ and Lemma 4.1.4 we obtain for $t \in [0,T], t \neq t_k$, that

$$\begin{aligned} p'_{n+1}(t) &= f(t,x) - f(t,\alpha_n) + g(t,x) - g(t,\beta_n) - f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &\quad - g_x(t,\alpha_n)(\beta_{n+1} - \beta_n) \\ &\leq f_x(t,\alpha_n)p_{n+1} - g_x(t,\alpha_n)q_{n+1} + [f_x(t,x) - f_x(t,\alpha_n)]p_n \\ &\quad + [g_x(t,\alpha_n) - g_x(t,\beta_n)]q_n \end{aligned}$$

$$\leq Q^{(1)}p_{n+1} + Q^{(2)}q_{n+1} + L^{(1)}p_n||p_n|| + L^{(2)}q_n||\alpha_n - \beta_n||, \tag{4.31}$$

where $Q^{(l)} = \left(Q_{ij}^{(l)}\right)_{i,j=1}^{n}$, $|\frac{\partial f_i(t,x)}{\partial x_j}| \leq Q_{ij}^{(1)}$, $|\frac{\partial g_i(t,x)}{\partial x_j}| \leq Q_{ij}^{(2)}$ for $(t,x) \in \Omega(\alpha_0,\beta_0),\ i,j = 1,2,\ldots,n,\ l = 1,2.$

We note that

$$||\alpha_n - \beta_n|| \leq ||p_n|| + ||q_n||$$

and

$$L^{(1)}p_n \leq l_1||p_n||, L^{(2)}q_n \leq l_2||q_n||,$$

where $l_j = (l_{j1}, l_{j2}, \ldots, l_{jn}), j = 1,2$ and $l_{ji} = max\{L_{im}^{(j)} :\ 1 \leq m \leq n\}$.

Therefore, for $t \in [0,T], t \neq t_k$ we obtain from (4.31) the inequality

$$p'_{n+1}(t) \leq Q^{(1)}p_{n+1}(t) + Q^{(2)}q_{n+1}(t) + l_1||p_n||^2 + l_2||q_n||^2 + l_2||p_n||.||q_n||. \tag{4.32}$$

Similarly, we obtain that for $t \in [0,T], t \neq t_k$

$$\begin{aligned} q'_{n+1}(t) &= f(t,\beta_n) - f(t,x) + g(t,\alpha_n) - g(t,x) \\ &\quad + f_x(t,\alpha_n)(\beta_{n+1} - \beta_n) + g_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &\leq f_x(t,\alpha_n)q_{n+1} - g_x(t,\alpha_n)p_{n+1} + L^{(1)}q_n||q_n + p_n|| + L^{(2)}p_n||p_n|| \end{aligned}$$

$$\leq Q^{(1)}q_{n+1}(t) + Q^{(2)}p_{n+1}(t) + l_1||q_n||.||q_n + p_n|| + l_2||p_n||^2. \tag{4.33}$$

We note that from the inequality $ab \leq (a^2 + b^2)/2$ it follows

$$||q_n||.||q_n + p_n|| \leq ||q_n||^2 + ||q_n||.||p_n|| \leq \frac{3}{2}||q_n||^2 + \frac{1}{2}||p_n||^2.$$

Therefore, for $t \in [0,T], t \neq t_k$ from inequalities(4.31), (4.32), (4.33), we obtain that

$$p'_{n+1}(t) \leq Q^{(1)}p_{n+1}(t) + Q^{(2)}q_{n+1}(t) + (l_1 + \frac{1}{2}l_2)||p_n||^2 + \frac{3}{2}l_2||q_n||^2,$$

$$q'_{n+1}(t) \leq Q^{(1)}q_{n+1}(t) + Q^{(2)}p_{n+1}(t) + \frac{3}{2}l_1||q_n||^2 + (\frac{1}{2}l_1 + l_2)||p_n||^2. \tag{4.34}$$

We can write the differential inequalities (4.34) in a vector form

$$\xi'_{n+1}(t) \leq Q\xi_{n+1}(t) + N_n(t), \quad t \in [0,T], t \neq t_k, \tag{4.35}$$

where

$$\xi_{n+1}(t) = \begin{pmatrix} p_{n+1}(t) \\ q_{n+1}(t) \end{pmatrix}, Q = \begin{pmatrix} Q^{(1)} & Q^{(2)} \\ Q^{(2)} & Q^{(1)} \end{pmatrix},$$

$$N_n(t) = \begin{pmatrix} (l_1 + \frac{1}{2}l_2)||p_n(t)||^2 & (\frac{3}{2}l_2)||q_n(t)||^2 \\ (\frac{3}{2}l_1)||q_n(t)||^2 & (l_2 + \frac{1}{2}l_1)||p_n(t)||^2 \end{pmatrix}.$$

Analogously it can be proved that there exist matrices $A_{k,n}$ and $B_{k,n}$, $(k = 1, 2, \ldots, p)$ with nonnegative elements, such that the inequalities

$$\xi_{n+1}(t_k + 0) \leq A_{k,n}\xi_{n+1}(t_k) + B_{k,n} \tag{4.36}$$

hold.

From inequalities (4.35), (4.36), and equality $\xi_{n+1}(0) = 0$ according to Lemma 4.1.5 follows that

$$\xi_{n+1}(t) \leq \int_0^t W_n(t;\eta)N_n(\eta)d\eta + \sum_{0<t_k<t} W_n(t;t_k)B_{k,n} \ , \tag{4.37}$$

where $W_n(t;s) = e^{Q(t-s)}\prod_{s<t_j<t}(E + A_{j,n}) \leq e^{QT}\prod_{j=1}^{p}(E + A_{j,n})$.

Inequality (4.37) implies that there exist constants $\lambda_i \geq 0, i = 1,2,3,4$ such that

$$|||p_{n+1}||| \leq \lambda_1|||p_n|||^2 + \lambda_2|||q_n|||^2 \tag{4.38}$$

and

$$|||q_{n+1}||| \leq \lambda_3|||p_n|||^2 + \lambda_4|||q_n|||^2, \tag{4.39}$$

where $|||u||| = sup\{||u(t)|| : t \in [0,T]\}$.

Inequalities (4.38) and (4.39) imply that the convergence is quadratic. □

In the case, when the derivatives of the functions $f(t,x)$ and $g(t,x)$ are nonincreasing and nondecreasing, respectively, the following result is valid.

Theorem 4.1.2. *Let the following conditions be fulfilled:*

1. *Functions* $\alpha_0, \beta_0 \in PC_1([0,T], \mathbf{R}^n)$ *form a mixed couple of lower and upper solutions of the initial value problem* (4.1), (4.2), (4.3) *and* $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$.

2. *Functions* $f, g \in C([0,T] \times \mathbf{R}^n, \mathbf{R}^n)$ *and the derivatives* $f_x(t,x), g_x(t,x)$ *exist and they are continuous on* $\Omega(\alpha_0, \beta_0)$, $f_x(t,x)$ *is nonincreasing in* x , $g_x(t,x)$ *is nondecreasing in* x *for* $t \in [0,T]$ *and*

$$f_x(t,x) - f_x(t,y) \leq L^{(1)}||x-y||, \quad g_x(t,x) - g_x(t,y) \leq L^{(2)}||x-y||$$

where $L^{(1)}, L^{(2)}$ *are constant matrices with positive elements.*

3. *Derivatives* $I_k'(x), J_k'(x)$, $(k = 1,2,\ldots,p)$ *exist and they are continuous on the sets* $\Gamma_k(\alpha_0,\beta_0)$*, derivatives* $I_k'(x)$ *are nonincreasing, derivatives* $J_k'(x)$ *are nondecreasing,* $I_k'(x) \geq 0 \geq J_k'(x)$*, and*

$$I_k'(x) - I_k'(y) \leq M_k^{(1)}||x-y||, \quad J_k'(x) - J_k'(y) \leq M_k^{(2)}||x-y||$$

for $x,y \in \Gamma_k(\alpha_0,\beta_0)$*, where* $M_k^{(1)}, M_k^{(2)}$ *are constant matrices with positive elements.*

4. *For each* $\eta \in S(\alpha_0,\beta_0)$ *the function* $f_x(t,\eta)x$ *is nondecreasing in x and the function* $\Theta(t,\eta,x) = g_x(t,\eta)x$ *is nonincreasing in x.*

Then there exist two sequences $\{\alpha_n(t)\}_0^\infty$ *and* $\{\beta_n(t)\}_0^\infty$ *such that*

a/ The sequences are monotone increasing and monotone decreasing, respectively;

b/ The functions $\alpha_n(t), \beta_n(t)$ *form a mixed couple of upper and lower solutions of the initial value problem* (4.1), (4.2), (4.3);

c/ Both sequences converge uniformly to the unique solution of the initial value problem (4.1), (4.2), (4.3) *in* $S(\alpha_0,\beta_0)$ *for* $t \in (t_k,t_{k+1}]$, $k = 0,1,2,\ldots,p$, $t_0 = 0$, $t_{p+1} = T$;

d/ The convergence is quadratic, i.e. there exist constants $\lambda_i \geq 0, i = 1,2,3,4$ *such that*

$$|||x(t) - \alpha_{n+1}(t)||| \leq \lambda_1|||x(t) - \alpha_n(t)|||^2 + \lambda_2|||\beta_n(t) - x(t)|||^2$$

and

$$|||\beta_{n+1}(t) - x(t)||| \leq \lambda_3|||x(t) - \alpha_n(t)|||^2 + \lambda_4|||\beta_n(t) - x(t)|||^2,$$

where $|||u||| = sup\{||u(t)|| : t \in [0,T]\}$.

Proof. For each $n = 0,1,2,\ldots$ we consider the following initial value problem for the linear impulsive differential equations

$$\begin{aligned} x' &= f(t,\alpha_n) + f_x(t,\beta_n)(x-\alpha_n) + g(t,\beta_n) + g_x(t,\beta_n)(y-\beta_n) \\ y' &= f(t,\beta_n) + f_x(t,\beta_n)(y-\beta_n) + g(t,\alpha_n) + g_x(t,\beta_n)(x-\alpha_n), \\ &\text{for } t \neq t_k, \end{aligned} \tag{4.40}$$

$$\begin{aligned} x(t_k+0) &= I_k(\alpha_n(t_k)) + I_k'(\beta_n(t_k))(x(t_k) - \alpha_n(t_k)) \\ &\quad + J_k(\beta_n(t_k)) + J_k'(\beta_n(t_k))(y(t_k) - \beta_n(t_k)), \\ y(t_k+0) &= I_k(\beta_n(t_k)) + I_k'(\beta_n(t_k))(y(t_k) - \beta_k(t_k)) \\ &\quad + J_k(\alpha_n(t_k)) + J_k'(\beta_n(t_k))(x(t_k) - \alpha_n(t_k)), \\ &k = 1,2,\ldots,p, \end{aligned} \tag{4.41}$$

$$x(0) = x_0 = y(0). \tag{4.42}$$

The rest part of the proof is analogous to the proof of Theorem 4.1.1. □

Remark 19. We note that in the scalar case $N = 1$, the obtained results concern to the initial value problem for scalar impulsive differential equations ([120]). On the other side, the obtained results are different from the results in [38] and the conditions in Theorem 4.1.1 and Theorem 4.1.2 are more practically useful.

Remark 20. Presenting the right parts of the impulsive differential equations as sums of two functions, which derivatives are increasing or decreasing, gives us the possibilities of application of quasilinearization to wider class of impulsive differential equations.

4.2. Method of Quasilinearization for a Linear Boundary Value Problem for Scalar Impulsive Differential Equations

In this section impulsive differential equations with a linear two point boundary condition are studied. An existence theorem is proved. An algorithm, based on the method of quasilinearization, for constructing successive approximations of the solution of the considered problem is given. The quadratic convergence of the iterates is proved.

Consider the system of the nonlinear impulsive differential equation

$$x' = f(t, x(t)) \quad \text{for} \quad t \in [0,T], \; t \neq t_k, \tag{4.43}$$

$$x(t_k + 0) = I_k(x(t_k)), \quad k = 1,2,\dots,p \tag{4.44}$$

with the linear boundary value condition

$$Mx(0) - Nx(T) = c, \tag{4.45}$$

where $x \in \mathbf{R}$, $f : [0,T] \times \mathbf{R} \to \mathbf{R}$, $I_k : \mathbf{R} \to \mathbf{R}$, $(k = 1,2,\dots,p)$, c, M, N are constants.

Definition 28. The function $\alpha(t) \in PC^1([0,T],\mathbf{R})$ is called a lower (upper) solution of the boundary value problem (4.43), (4.44), (4.45), if the inequalities

$$\alpha'(t) \leq (\geq) \; f(t, \alpha(t)) \quad \text{for} \quad t \in [0,T], t \neq t_k, \tag{4.46}$$

$$\alpha(t_k + 0) \leq (\geq) \; I_k(\alpha(t_k)), \quad k = 1,2,\dots,p, \tag{4.47}$$

$$M\alpha(0) - N\alpha(T) \leq (\geq) c \tag{4.48}$$

hold.

Let functions $\alpha, \beta \in PC([0,T],\mathbf{R})$ be such that $\alpha(t) \leq \beta(t)$.

Consider the linear boundary value problem for the linear impulsive differential equations

$$u'(t) = g(t)u(t) + \sigma(t), \; t \in [0,T], \; t \neq t_k, \tag{4.49}$$

$$u(t_k + 0) = b_k u(t_k) + \gamma_k, \; k = 1,2,\dots,p, \tag{4.50}$$

$$Mu(0) - Nu(T) = 0. \tag{4.51}$$

Using the results for the initial value problem for the linear impulsive differential equations (corollary 1.6.1 [89]), we could prove that the boundary value problem (4.49), (4.50), (4.51) has an unique solution.

Lemma 4.2.1. *Let functions $g, \sigma \in PC([0,T],\mathbf{R})$ and $M, N, b_k, \gamma_k, \; (k = 1,2,\dots p)$ be constants such that $N(\prod_{k=1}^{p} b_k) exp\Big(\int_0^T g(s)ds\Big) \neq M$.*

Then the boundary value problem (4.49), (4.50), (4.51) *has an unique solution* $u(t)$ *on* $[0,T]$, *where*

$$\begin{aligned} u(t) &= u(0)\Big(\prod_{0<t_k<t} b_j\Big)exp(\int_0^t g(\tau)d\tau) \\ &+ \sum_{0<t_k<t}\gamma_k\Big(\prod_{t_k<t_j<t} b_j\Big)exp(\int_{t_k}^t g(\tau)d\tau)ds \\ &+ \int_0^t \sigma(s)\Big(\prod_{s<t_k<t} b_k\Big)exp(\int_s^t g(\tau)d\tau)ds, \end{aligned}$$

$$b_0 = 1,\ \prod_{j=k}^{n} f(j) = 1 \text{ for } k > n,$$

$$\begin{aligned} u(0) &= \Big[1 - \frac{N}{M}\Big(\prod_{k=1}^{p} b_k\Big)exp\Big(\int_0^T g(s)ds\Big)\Big]^{-1} \\ &\times\Big\{\sum_{i=1}^{p}\gamma_i\Big(\prod_{j=i+1}^{p} b_j\Big)exp(\int_{t_i}^T g(\tau)d\tau)ds \\ &+ \int_0^T \sigma(s)\Big(\prod_{s<t_j<T} b_j\Big)exp(\int_s^T g(\tau)d\tau)\Big\}. \end{aligned}$$

We will use the following result for impulsive differential inequalities:

Lemma 4.2.2 (Theorem 1.4.1 [89]). *Let function* $m \in PC^1([0,\infty),\mathbf{R})$ *and let the inequalities be satisfied*

$$\begin{aligned} &m'(t) \le p(t)m(t) + q(t),\ \ t \ge 0,\ t \ne t_k, \\ &m(t_k+0) \le d_k m(t_k) + b_k,\ k = 1,2,\ldots, \end{aligned}$$

where $d_k,\ b_k\ \ (k = 1,2,\ldots,p)$ *are constants,* $d_k \ge 0$, $p,q \in C([0,\infty),\mathbf{R})$.

Then the inequality

$$\begin{aligned} m(t) &\le m(0)\Big(\prod_{0<t_k<t} d_j\Big)exp(\int_0^t p(\tau)d\tau) \\ &+ \sum_{0<t_k<t} b_k\Big(\prod_{t_k<t_j<t} d_j\Big)exp(\int_{t_k}^t p(\tau)d\tau) \\ &+ \int_0^t \Big(\prod_{s<t_k<t} d_k\Big)exp(\int_s^t p(\tau)d\tau)q(s)ds \end{aligned}$$

hold for $t \ge 0$.

The following comparisons result will be used in the proof of the main result in this section.

Lemma 4.2.3. *Let function* $m \in PC^1([0,T],\mathbf{R})$ *and the inequalities be satisfied*

$$m'(t) \le \phi(t)m(t), \quad t \in [0,T], t \ne t_k, \tag{4.52}$$

$$m(t_k+0) \le \alpha_k m(t_k), \quad k = 1,2,\dots,p \tag{4.53}$$

$$Mm(0) - Nm(T) \le 0, \tag{4.54}$$

where $\alpha_k \ge 0, M > 0, N \ge 0$ *are constants such that*

$$M - N\Big(\prod_{k=1}^{p}\alpha_k\Big)exp\Big(\int_0^T \phi(s)ds\Big) > 0. \tag{4.55}$$

Then $m(t) \le 0$ *for* $t \in [0,T]$.

Proof. Function $m(t)$ satisfies the integral inequality

$$m(t) \le m(0) + \int_0^t \phi(s)m(s)ds + \sum_{k:0<t_k<t}\alpha_k m(t_k).$$

According to Theorem 1.1.1 function $m(t)$ satisfies the inequality

$$m(t) \le m(0)\Big(\prod_{k:0<t_k<t}\alpha_k\Big)exp\Big(\int_0^t \phi(s)ds\Big) \text{ for } t \in [0,T]. \tag{4.56}$$

From inequality (4.54) we obtain that

$$m(0) \le \frac{N}{M}m(T)$$

and therefore

$$m(0) \le \frac{N}{M}m(0)\Big(\prod_{k=1}^{p}\alpha_k\Big)exp(\int_0^T \phi(s)ds). \tag{4.57}$$

From inequalities (4.55) and (4.57) follows that $m(0) \le 0$. From (4.56) follows the validity of the inequality $m(t) \le 0$ for $t \in [0,T]$. □

As a partial case of Lemma 4.2.3 we obtain the following result:

Corollary 4.2.7. *Let function* $m \in PC^1([0,T],\mathbf{R})$ *and inequalities* (4.52)–(4.54) *be satisfied, where* $\int_0^T \phi(s)ds \le 0$, $0 \le \alpha_k < 1$ *and* $M > 0,\ N \ge 0,\ M \ge N$.

Then function $m(t)$ *is nonpositive on* $[0,T]$.

We will obtain sufficient conditions for the existence of a solution of the linear boundary value problem for the nonlinear impulsive differential equations (4.43), (4.44), (4.45), that is between given lower and upper solutions. The obtained result will be useful not only for the proof of the method of quasilinearization but for different qualitative investigations of nonlinear boundary value problem for impulsive differential equations.

Theorem 4.2.1. *Let the following conditions be fulfilled:*

1. *Functions* $\alpha,\beta \in PC^1([0,T],\mathbf{R})$ *are lower and upper solutions of the linear boundary value problem* (4.43), (4.44), (4.45) *and* $\alpha(t) \le \beta(t)$ *for* $t \in [0,T]$.

2. *Function* $f \in C(\Omega(\alpha,\beta),\mathbf{R})$, *where the set* $\Omega(\alpha,\beta)$ *is defined by equality* (4.8) *for* $N = 1$.

3. *Functions* $I_k : \Gamma_k(\alpha,\beta) \to \mathbf{R}, (k = 1,2,\dots)$ *are nondecreasing in* $\Gamma_k(\alpha,\beta)$, *where the sets* $\Gamma_k(\alpha,\beta)$ *are defined by equality* (4.9) *for* $N = 1$.

4. *Constants* $M > 0, N \ge 0$.

Then the linear boundary value problem for nonlinear impulsive differential equations (4.43), (4.44), (4.45) *has a solution* $u \in S(\alpha,\beta)$, *where the set* $S(\alpha,\beta)$ *is defined by* (4.7) *for* $N = 1$.

Proof. Without loss of generality we will consider the case when $p = 1$, i.e. $0 < t_1 < T$. Let x_0 be an arbitrary point such that $\alpha(0) \le x_0 \le \beta(0)$. Define a function $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ by the equality

$$F(t,x) = \begin{cases} f(t,\beta(t)) + \frac{\beta(t)-x}{1+|x|} & \text{for } x > \beta(t) \\ f(t,x) & \text{for } \alpha(t) \le x \le \beta(t) \\ f(t,\alpha(t)) + \frac{\alpha(t)-x}{1+|x|} & \text{for } x < \alpha(t). \end{cases}$$

From condition 2 of Theorem 4.2.1 follows that function $f(t,x)$ is bounded on $\Omega(\alpha,\beta)$ and therefore there exists a function $\mu \in C([0,T],[0,\infty))$ such that $sup\{|F(t,x)| : x \in \mathbf{R}\} \le \mu(t)$ for $t \in [0,T]$.

Therefore, the initial value problem for the ordinary differential equation $x' = F(t,x)$, $x(0) = x_0$ has a solution $X(t;x_0)$ for $t \in [0,t_1]$.

Consider function $m(t) = X(t;x_0) - \beta(t)$. We will prove that function $m(t)$ is nonpositive on $[0,t_1]$. Assume the opposite, i.e. $sup\{m(t) : t \in [0,t_1]\} > 0$. Therefore, there exists a point $t^* \in (0,t_1)$ such that $m(t^*) > 0$ and $m'(t^*) \ge 0$. From the definition of the function $X(t;x_0)$ it also follows that

$$m'(t^*) \le f(t^*,\beta(t^*)) + \frac{\beta(t^*) - X(t^*;x_0)}{1+|X(t^*;x_0)|} - f(t^*,\beta(t^*)) = \frac{-m(t^*)}{1+|X(t^*;x_0)|} < 0.$$

The obtained contradiction proves the assumption is false. Therefore

$$X(t;x_0) \le \beta(t), \quad t \in [0,t_1].$$

Analogously, we can prove that $X(t;x_0) \ge \alpha(t)$, $t \in [0,t_1]$.

Let $y_0 = I_1(X(t_1;x_0))$. We note that y_0 depends on x_0. From the monotonicity of the function $I_1(x)$ we obtain

$$\alpha(t_1+0) \quad \le \quad I_1(\alpha(t_1)) \le I_1(X(t_1;x_0)) \le I_1(\beta(t_1)) \le \beta(t_1+0),$$

i.e.

$$\alpha(t_1+0) \le y_0 \le \beta(t_1+0).$$

Consider the initial value problem for the ordinary differential equation $x' = F(t,x)$, $x(t_1) = y_0$ for $t \in [t_1,T]$. This initial value problem has a solution $Y(t;y_0)$ for $t \in [t_1,T]$.

Using the same ideas as above we can prove that the inequalities $\alpha(t) \leq Y(t;y_0) \leq \beta(t)$ for $t \in [t_1, T]$ hold. At the same time $Y(t_1;y_0) = I_1(X(t_1;x_0))$.

Define the following function

$$x(t;x_0) = \begin{cases} X(t;x_0) & \text{for } t \in [0,t_1] \\ Y(t;y_0) & \text{for } t \in (t_1,T]. \end{cases}$$

Function $x(t;x_0) \in S(\alpha,\beta)$ is a solution of the impulsive differential equation (4.43), (4.44) with the initial condition $x(0) = x_0$.

From inequality $\alpha(t) \leq \beta(t)$ for $t \in [0,T]$ it follows that the following two cases are possible:

Case 1. Let $\alpha(0) = \beta(0)$. Then $x_0 = \alpha(0) = \beta(0)$. Therefore

$$Mx(0;x_0) - Nx(T;x_0) = Mx_0 - Nx(T;x_0) \leq M\alpha(0) - N\alpha(T) \leq c$$

and

$$Mx(0;x_0) - Nx(T;x_0) \geq Mx_0 - N\beta(T) \geq c.$$

Therefore, $Mx(0;x_0) - Nx(T;x_0)) = c$, i.e. the function $x(t;x_0)$ is a solution of the linear boundary value problem (4.43), (4.44), (4.45).

Case 2. Let $\alpha(0) < \beta(0)$. We will prove that there exists a point $x_0 \in [\alpha(0), \beta(0)]$ such that the solution $x(t;x_0)$ of the impulsive differential equation (4.43), (4.44) with the initial condition $x(0) = x_0$ satisfies the boundary condition (4.45).

Assume the opposite, i.e. for every point $x_0 \in [\alpha(0), \beta(0)]$ the solution $x(t;x_0)$ of the impulsive equation (4.43), (4.44) with the initial condition $x(0;x_0) = x_0$ satisfies the inequality $Mx(0;x_0) - Nx(T;x_0)) \neq c$.

If $x_0 = \beta(0)$ then from the relation $x(t;x_0) \in S(\alpha,\beta)$ we obtain that

$$Mx(0;x_0) - Nx(T;x_0)) = M\beta(0) - Nx(T;x_0) \geq M\beta(0) - N\beta(T) \geq c.$$

According to the assumption and the above inequality we obtain

$$Mx(0;x_0) - Nx(T;x_0) > c. \tag{4.58}$$

There exists a number $\delta : 0 < \delta < \beta(0) - \alpha(0)$, such that for $x_0 : 0 \leq \beta(0) - x_0 < \delta$ the corresponding solution $x(t;x_0)$ of the impulsive differential equation (4.43), (4.44) satisfies the inequality

$$Mx(0;x_0) - Nx(T;x_0) > c. \tag{4.59}$$

Assume that for every natural number n there exists a point $z_n : 0 \leq \beta(0) - z_n < \frac{1}{n}$ such that the corresponding solution $x^{(n)}(t;z_n)$ of the impulsive equation (4.43), (4.44) with initial condition $x(0) = z_n$ satisfies the inequality

$$Mx^{(n)}(0;z_n) - Nx^{(n)}(T;z_n) < c.$$

Let $\{z_{n_j}\}$ be a subsequence of the sequence $\{z_n\}_{n=1}^{\infty}$ such that $lim_{j\to\infty} z_{n_j} = \beta(0)$ and $lim_{j\to\infty} x^{(n_j)}(t;z_{n_j}) = x(t)$ uniformly on the intervals $[0,t_1]$ and $(t_1,T]$. The function $x(t)$

is a solution of the impulsive differential equation (4.43), (4.44) such that $x(0) = \beta(0)$, $x(t) \in S(\alpha,\beta)$ and

$$Mx(0) - Nx(T) \leq c. \tag{4.60}$$

Inequality (4.60) contradicts inequality (4.57) and therefore the assumption is not true. Let

$$\begin{aligned}\delta^* = \; & sup\{\delta \in (0,\beta(0)-\alpha(0)] : \textit{ for which there exists a point} \\ & x_0 \in (\beta(0)-\delta,\beta(0)] \textit{ such that the solution } x(t;x_0) \\ & \textit{satisfies the inequality (4.59)}\}.\end{aligned}$$

Choose a sequence of points $x_n \in (\alpha(0),\beta(0)-\delta^*)$ such that $\lim_{n\to\infty} x_n = \beta(0)-\delta^*$. From the choice of δ^* and the assumption it follows that the corresponding solutions $x^{(n)}(t;x_n)$ satisfy the inequality

$$Mx^{(n)}(0;x_n) - Nx^{(n)}(T;x_n) < c.$$

There exists a subsequence $\{x_{n_j}\}_0^\infty$ of the sequence $\{x_n\}_0^\infty$ such that the limit $lim_{j\to\infty}\, x^{(n_j)}(t;x_{n_j}) = x^*(t)$ uniformly on the intervals $[0,t_1]$ and $(t_1,T]$. Function $x^*(t) \in S(\alpha,\beta)$ is a solution of the impulsive equation (4.43), (4.44) with initial condition $x(0) = \beta(0)-\delta^*$ and satisfies the inequality $Mx^*(0) - Nx^*(T) \leq c$. The last inequality contradicts the choice of δ^*.

Therefore, there exists a point $x_0 \in [\alpha(0),\beta(0)]$ such that the solution $x(t;x_0)$ of the impulsive differential equation (4.43), (4.44) satisfies the condition (4.45), i.e. the function $x(t;x_0)$ is a solution of the linear boundary value problem (4.43), (4.44), (4.45). This completes the proof of the theorem. □

We will apply the method of quasilinearization to approximate the solution of the linear boundary value problem (4.43), (4.44), (4.45). We will prove that the convergence of the successive approximations is quadratic.

Theorem 4.2.2. *Let the following conditions hold:*

1. *Functions* $\alpha_0(t),\beta_0(t)$ *are lower and upper solutions of the linear boundary value problem* (4.43), (4.44), (4.45) *and* $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$.

2. *Function* $f \in C^{0,2}(\Omega(\alpha_0,\beta_0),\mathbf{R})$ *and there exist two functions* $F(t,x), g(t,x)$ *such that*

$$F, g \in C^{0,2}(\Omega(\alpha_0,\beta_0),\mathbf{R})$$

$$F(t,x) = f(t,x) + g(t,x),\;\; F''_{xx}(t,x) \geq 0,\;\; g''_{xx}(t,x) \geq 0,$$

$$\int_0^T [F'_x(s,\beta_0(s)) - g'_x(s,\alpha_0(s))]ds < 0.$$

3. *Functions* $I_k \in C^2(\Gamma_k(\alpha_0,\beta_0),\mathbf{R})$, $k = 1,2,\dots,p$ *and there exist functions* $G_k, J_k \in C^2(\Gamma_k(\alpha_0,\beta_0),\mathbf{R})$ *such that* $G_k(x) = I_k(x) + J_k(x)$, *the derivatives* $G''_k(x) \geq 0$, $J''_k(x) \geq 0$,

$$G'_k(\beta_0(t_k)) - J'_k(\alpha_0(t_k)) < 1,\;\; k = 1,2,\dots,p,$$

$$G'_k(\alpha_0(t_k)) - J'_k(\beta_0(t_k)) \geq 0,\;\; k = 1,2,\dots,p.$$

4. *Constants* $M > 0, N \geq 0$, $M \geq N$.

Then there exist two sequences of functions $\{\alpha_n(t)\}_0^\infty$ *and* $\{\beta_n(t)\}_0^\infty$ *such that:*

a/ The sequences are increasing and decreasing correspondingly;

b/ Functions $\alpha_n(t)$ *are lower solutions, and functions* $\beta_n(t)$ *are upper solutions of the linear boundary value problem* (4.43), (4.44), (4.45)*;*

c/Both sequences converge uniformly to the unique solution of the linear boundary value problem (4.43), (4.44), (4.45) *in* $S(\alpha_0,\beta_0)$ *for* $t \in (t_k,t_{k+1}]$, *where* $k = 0,1,2,\dots,p$, $t_0 = 0$, *and* $t_{p+1} = T$*;*

d/ The convergence is quadratic.

Proof. From condition 2 of Theorem 4.2.2 follows that if (t,x_1), $(t,x_2) \in \Omega(\alpha_0,\beta_0)$ and $x_1 \geq x_2$ then

$$f(t,x_1) \geq f(t,x_2)+F'_x(t,x_2)(x_1-x_2)+g(t,x_2)-g(t,x_1), \tag{4.61}$$

$$g(t,x_1) \geq g(t,x_2)+g'_x(t,x_2)(x_1-x_2). \tag{4.62}$$

From condition 3 of Theorem 4.2.2 follows that if $x_1 \geq x_2$, $x_1,x_2 \in \Gamma_k(\alpha_0,\beta_0)$, then

$$I_k(x_1) \geq I_k(x_2)+G'_k(x_2)(x_1-x_2)+J_k(x_2)-J_k(x_1), \tag{4.63}$$

and

$$G_k(x_1) \geq G_k(x_2)+G'_k(x_2)(x_1-x_2). \tag{4.64}$$

From condition 3 follows that functions $G'_k(x)$ and $J'_k(x)$ are nondecreasing in $D_k(\alpha_o,\beta_0)$. Therefore for $x \in D_k(\alpha_0,\beta_0)$ the inequality $I'_k(x) = G'_k(x) - J'_k(x) \geq G'_k(\alpha_0(t_k)) - J'_k(\beta_0(t_k)) \geq 0$ holds, which proves that the functions $I_k(x)$ are nondecreasing, $k = 1,2,\dots,p$.

According to Theorem 4.2.1 the boundary value problem (4.43), (4.44), (4.45) has a solution in $S(\alpha_0,\beta_0)$.

We consider the linear boundary value problem for the impulsive linear differential equation

$$x'(t) = f(t,\alpha_0(t))+Q_0(t)(x-\alpha_0(t)) \text{ for } t \in [0,T], t \neq t_k, \tag{4.65}$$

$$x(t_k+0) = I_k(\alpha_0(t_k))+B_k^0[x(t_k)-\alpha_0(t_0)], \tag{4.66}$$

$$Mx(0)-Nx(T) = c, \tag{4.67}$$

where

$$\begin{aligned} &Q_0(t) = F'_x(t,\alpha_0(t))-g'_x(t,\beta_0(t)),\\ &B_k^0 = G'_k(\alpha_0(t_k))-J'_k(\beta_0(t_k)), \quad k = 1,2,\dots,p. \end{aligned}$$

It is easy to verify that function $\alpha_0(t)$ is a lower solution of the linear boundary value problem (4.65), (4.66), (4.67).

According to condition 1 of Theorem 4.2.2 and inequalities (4.61) - (4.64) we obtain the inequalities

$$\begin{aligned} \beta'_0(t) \quad \geq \quad & f(t,\alpha_0(t))+Q_0(t)(\beta_0(t)-\alpha_0(t))\\ &-[F(t,\alpha_0(t))-F(t,\beta_0(t))+F'_x(t,\alpha_0(t))(\beta_0(t)-\alpha_0(t))]\\ &+g(t,\alpha_0(t))-g(t,\beta_0(t))+g'_x(t,\beta_0(t))(\alpha_0(t)-\beta_0(t)) \end{aligned}$$

$$\geq f(t,\alpha_0(t)) + Q_0(t)(\beta_0(t) - \alpha_0(t)) \quad \text{for} \quad t \in [0,T],\ t \neq t_k, \tag{4.68}$$

$$\begin{aligned} \beta_0(t_k+0) \quad \geq \quad & I_k(\alpha_0(t_k)) + [I_k(\beta_0(t_k)) - I_k(\alpha_0(t_k))] \\ & \geq I_k(\alpha_0(t_k)) + [G_k'(\alpha_0(t_k)) - J_k'(\beta_0(t_k))](\beta_0(t_k) - \alpha_0(t_k)) \\ & \geq I_k(\alpha_0(t_k)) + B_k^0(\beta_0(t_k) - \alpha_0(t_k)). \end{aligned} \tag{4.69}$$

From inequalities (4.68) and (4.69) follows that function $\beta_0(t)$ is an upper solution of the linear boundary value problem (4.65), (4.66), (4.67).

According to Lemma 4.2.1 the linear boundary value problem (4.65), (4.66), (4.67) has a unique solution $\alpha_1(t) \in S(\alpha_0, \beta_0)$.

We consider the linear boundary value problem for the impulsive linear differential equation

$$x'(t) = f(t,\beta_0(t)) + Q_0(t)(x(t) - \beta_0(t)) \quad \text{for} \quad t \in [0,T], t \neq t_k, \tag{4.70}$$

$$x(t_k+0) = I_k(\beta_0(t_k)) + B_k^0(x(t_k) - \beta_0(t_k)), \tag{4.71}$$

$$Mx(0) - Mx(T) = c. \tag{4.72}$$

Functions $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the linear boundary value problem (4.70), (4.71), (4.72) and according to Lemma 4.2.1 there exists a unique solution $\beta_1(t) \in S(\alpha_0, \beta_0)$.

We will prove that $\alpha_1(t) \leq \beta_1(t)$ for $t \in [0,T]$.

Define function $u(t) = \alpha_1(t) - \beta_1(t)$ for $t \in [0,T]$. From the choice of functions $\alpha_1(t)$ and $\beta_1(t)$ and inequality (4.70) we obtain that function $u(t)$ for $t \in [0,T], t \neq t_k$ satisfies the inequality

$$u' = f(t,\alpha_0(t)) - f(t,\beta_0(t)) + Q_0(t)u(t) + Q_0(t)(\beta_0(t) - \alpha_0(t)) \leq Q_0(t)u(t). \tag{4.73}$$

According to equalities (4.71) for $x_2 = \beta_0(t_k)$ and $x_1 = \alpha_0(t_k)$, and the definition of the functions α_1, β_1 we obtain

$$u(t_k+0) \leq I_k(\alpha_0(t_k)) - I_k(\beta_0(t_k)) + B_k^0 u(t_k) + B_k^0[\beta_0(t_k) - \alpha_0(t_k)] \leq B_k^0 u(t_k). \tag{4.74}$$

From the boundary value condition for functions α_1, β_1 and condition 4 we obtain the equality

$$Mu(0) - Nu(T) = M\alpha_1(0) - N\alpha_1(T) - (M\beta_1(0) - N\beta_1(T)) = c - c = 0. \tag{4.75}$$

From inequalities (4.73), (4.74) and boundary condition (4.75), according to Lemma 4.2.3 function $u(t)$ is nonpositive, i.e. $\alpha_1(t) \leq \beta_1(t)$.

Function $\alpha_1(t)$ is a lower solution of the boundary value problem for the nonlinear scalar impulsive differential equation (4.43), (4.44), (4.45). Indeed, for $t \in [0,T]$, $t \neq t_k$,

$$\begin{aligned} \alpha_1' \quad \leq \quad & f(t,\alpha_1(t)) + F_x'(t,\alpha_0(t))(\alpha_0(t) - \alpha_1(t)) \\ & -g(t,\alpha_0(t)) + g(t,\alpha_1(t)) + Q_0(t)(\alpha_1(t) - \alpha_0(t)) \\ & \leq f(t,\alpha_1(t)). \end{aligned} \tag{4.76}$$

From inequality (4.63) and the choice of function $\alpha_1(t)$ we obtain the inequalities

$$\begin{aligned}\alpha_1(t_k+0) &\leq I_k(\alpha_1(t_k)) + [G_k'(\alpha_0(t_k)) - J_k'(\alpha_1(t_k)) \\ &\quad -B_k^0](\alpha_0(t_k) - \alpha_1(t_k)) \\ &\leq I_k(\alpha_1(t_k)) - [J_k'(\alpha_0(t_k)) - J_k'(\beta_0(t_k))](\alpha_0(t_k) - \alpha_1(t_k)) \\ &\leq I_k(\alpha_1(t_k)), \quad k = 1,2,\dots,p. \end{aligned} \tag{4.77}$$

From inequalities (4.76), (4.77) and the boundary condition for function $\alpha_1(t)$ follows that function $\alpha_1(t)$ is a lower solution of the linear boundary value problem (4.43), (4.44), (4.45).

Analogously, it can be proved that function $\beta_1(t)$ is an upper solution of the linear boundary value problem (4.43), (4.44), (4.45).

We can construct two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$, such that $\alpha_n, \beta_n \in S(\alpha_{n-1}, \beta_{n-1})$. Function $\alpha_{n+1}(t)$ is the unique solution of the linear boundary value problem for the impulsive linear differential equation

$$x'(t) = f(t, \alpha_n(t)) + Q_n(t)(x - \alpha_n(t)) \text{ for } t \in [0,T], t \neq t_k, \tag{4.78}$$

$$x(t_k+0) = I_k(\alpha_n(t_k)) + B_k^n(x(t_k) - \alpha_n(t_k)), \tag{4.79}$$

$$Mx(0) - Nx(T) = c, \tag{4.80}$$

and function $\beta_{n+1}(t)$ is the unique solution of the linear boundary value problem for the impulsive linear differential equation

$$x'(t) = f(t, \beta_n(t)) + Q_n(t)(x - \beta_n(t)) \text{ for } t \in [0,T], t \neq t_k, \tag{4.81}$$

$$x(t_k+0) = I_k(\beta_n(t_k)) + B_k^n(x(t_k) - \beta_n(t_k)), \tag{4.82}$$

$$Mx(0) - Nx(T) = c, \tag{4.83}$$

where

$$Q_n(t) = F_x'(t, \alpha_n(t)) - g_x'(t, \beta_n(t)),$$
$$B_k^n = G_k'(\alpha_n(t_k)) - J_k'(\beta_n(t_k)).$$

As in the case $n = 0$ it can be proved that functions $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ are lower and upper solutions of the linear boundary value problem (4.43),(4.44),(4.45) and the inequalities

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \cdots \leq \beta_0(t) \tag{4.84}$$

hold.

Therefore, the sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ are uniformly bounded and equicontinuous on intervals $(t_k, t_{k+1}], k = 0,1,2,\dots,p$ and they are uniformly convergent.

Denote

$$\lim_{n\to\infty} \alpha_n(t) = u(t), \quad \lim_{n\to\infty} \beta_n(t) = v(t).$$

From the uniform convergence and the definition of functions $\alpha_n(t)$ and $\beta_n(t)$ follows that

$$\alpha_0(t) \leq u(t) \leq v(t) \leq \beta_0(t). \tag{4.85}$$

From the linear boundary value problems for the impulsive linear differential equations (4.78) - (4.80) and (4.81) - (4.83) we obtain that the functions $u(t)$ and $v(t)$ are solutions of the boundary value problem (4.43), (4.44), (4.45) and therefore $u(t) = v(t)$.

We will prove that the convergence is quadratic.

Let us define functions $a_{n+1}(t) = u(t) - \alpha_{n+1}(t)$ and $b_{n+1}(t) = \beta_{n+1}(t) - u(t),\ t \in [0,T]$. For $t \in [0,T], t \neq t_k$ we obtain the inequalities

$$\begin{aligned} a'_{n+1} &\leq Q_n(t)a_{n+1}(t) + [F'_x(t,u(t)) - g'_x(t,\alpha_n(t)) - Q_n(t)]a_n(t) \\ &= Q_n(t)a_{n+1}(t) + F''_{xx}(t,\xi_1)a_n^2(t) \\ &\quad + g''_{xx}(t,\eta_1)a_n(t)(\beta_n(t) - \alpha_n(t)), \end{aligned} \tag{4.86}$$

where $u(t) \leq \xi_1 \leq \alpha_n(t)$, $\alpha_n(t) \leq \eta_1 \leq \beta_n(t)$.

It is easy to verify the validity of the inequality

$$a_n(t)(\beta_n(t) - \alpha_n(t)) = a_n(t)(b_n(t) + a_n(t)) \leq \frac{1}{2}b_n^2(t) + \frac{3}{2}a_n^2(t). \tag{4.87}$$

From inequalities (4.86) and (4.87) follows that for $t \in [0,T], t \neq t_k$ the inequality

$$a'_{n+1}(t) \leq Q_n(t)a_{n+1}(t) + \sigma_n(t), \tag{4.88}$$

holds, where

$$\sigma_n(t) = [F''_{xx}(t,\xi_1) + \frac{3}{2}g''_{xx}(t,\eta_1)]a_n^2 + \frac{1}{2}g''_{xx}(t,\eta_1)b_n^2.$$

Analogously, it can be proved that

$$a_{n+1}(t_k + 0) \leq B_k^n a_{n+1}(t_k) + \gamma_k, \tag{4.89}$$

where

$$\gamma_k = [G''_k(\omega_k) + \frac{3}{2}J''_k(\nu_k)]a_n^2(t_k) + \frac{1}{2}J''_k(\nu_k)b_n^2(t_k),$$

$$\alpha_n(t_k) \leq \omega_k \leq u(t_k), \alpha_n(t_k) \leq \nu_k \leq \beta_n(t_k), k = 1,2,\ldots,p.$$

From boundary conditions for functions $u(t)$ and $\alpha_n(t)$ we obtain the equality

$$Ma_{n+1}(0) - Na_{n+1}(T) = 0. \tag{4.90}$$

From inequalities (4.88) and (4.89) according to Lemma 4.2.2 follows that the function $a_{n+1}(t)$ satisfies the estimate

$$\begin{aligned} a_{n+1}(t) &\leq a_{n+1}(0)\Big(\prod_{0<t_k<t} B_k^n\Big) exp(\int_0^t Q_n(\tau)d\tau) \\ &\quad + \sum_{0<t_k<t} \gamma_k \Big(\prod_{t_k<t_j<t} B_j^n\Big) exp(\int_{t_k}^t Q_n(\tau)d\tau) \\ &\quad + \int_0^t \sigma_n(s)\Big(\prod_{s<t_k<t} B_k^n\Big) exp(\int_s^t Q_n(\tau)d\tau)ds. \end{aligned} \tag{4.91}$$

From the boundary condition (4.90) we have $a_{n+1}(0)=\frac{N}{M}a_{n+1}(T)$ and therefore

$$\begin{aligned} a_{n+1}(0) \quad \le \quad & \Big[1-\frac{N}{M}\Big(\prod_{k=1}^{p}B_k^n\Big)exp\Big(\int_0^T Q_n(s)ds\Big)\Big]^{-1} \\ & \times\Big\{\sum_{i=0}^{p}\gamma_i\Big(\prod_{j=i+1}^{p}B_j^n exp\Big(\int_{t_i}^T Q_n(\tau)d\tau\Big) \\ & +\int_0^T \sigma_n(s)\Big(\prod_{s<t_j<T}B_j^n\Big)exp\Big(\int_s^T Q_n(\tau)d\tau\Big)ds. \end{aligned} \tag{4.92}$$

From the properties of functions $F(t,x)$ and $g(t,x)$, the definition of $\sigma_n(t)$ and inequalities (4.91), (4.92) follows that there exist constants $\lambda_1>0$ and $\lambda_2>0$ such that

$$||a_{n+1}||\le\lambda_1||a_n||^2+\lambda_2||b_n||^2. \tag{4.93}$$

Analogously it can be proved that there exist constants $\mu_1>0$ and $\mu_2>0$ such that

$$||b_{n+1}||\le\mu_1||b_n||^2+\mu_2||a_n||^2. \tag{4.94}$$

Inequalities (4.93) and (4.94) prove that the convergence is quadratic. □

Remark 21. In the case when the constant $N=0$ in the boundary condition, the boundary value problem (4.43), (4.44), (4.45) is reduced to the initial value problem for impulsive differential equations for which the quasilinearization is applied in [42], [91], [118].

We also note that some of the results for ordinary differential equations, obtained in [91] are partial cases of the obtained results when $I_k(x)=x$.

4.3. Method of Quasilinearization for Periodic Boundary Value Problem for Systems of Impulsive Differential Equations

In this section the periodic boundary value problem for a system of nonlinear impulsive differential equations is studied. Different types of couples of lower and upper solutions are considered. Several results for systems of ordinary differential equations are obtained as partial cases of the proved theorems.

We will note some qualitative investigations of periodic boundary value problems for impulsive equations are obtained in [**?**], [42], [43], [46], [47], [48], [49], [50], [51].

Consider the periodic boundary value problem for the system of nonlinear impulsive differential equations

$$x'=f(t,x(t))+g(t,x(t)) \quad \text{for} \quad t\in[0,T],\ t\ne t_k, \tag{4.95}$$

$$x(t_k+0)=I_k(x(t_k))+G_k(x(t_k)),\ \ k=1,2,\dots,p, \tag{4.96}$$

$$x(0)=x(T), \tag{4.97}$$

where $x\in\mathbf{R}^N,\ f,g:[0,T]\times\mathbf{R}^N\to\mathbf{R}^N$, $I_k,G_k:\mathbf{R}^N\to\mathbf{R}^N$, $(k=1,2,\dots,p)$.

Sets $S(\alpha,\beta)$, $\Omega(\alpha,\beta)$ and $\Gamma_i(\alpha,\beta)$, $(i=1,2\dots,p)$, are defined by equalities (4.7), (4.8), (4.9).

We will define different types of lower and upper solutions of the considered periodic boundary value problem.

Definition 29. Function $\alpha_0(t) \in PC^1([0,T],\mathbf{R}^N)$ is called lower (upper) solution of the periodic boundary value problem (4.95), (4.96), (4.97), if the following inequalities

$$\begin{aligned}
&\alpha_0'(t) \leq (f(t,\alpha_0(t)) + g(t,\alpha_0(t)) \text{ for } t \in [0,T],\ t \neq t_k,\\
&\alpha_0(t_k+0) \leq I_k(\alpha_0(t_k)) + G_k(\alpha_0(t_k)),\ \ k = 1,2,\ldots,p,\\
&\alpha_0(0) \leq \alpha_0(T)
\end{aligned}$$

hold.

The above definition gives the natural definition for lower and upper solutions.

Definition 30. The pair of functions $\alpha_0(t),\beta_0(t) \in PC^1([0,T],\mathbf{R}^N)$ is called a *first type of mixed pair* of lower and upper solution of the periodic boundary value problem for the nonlinear system of impulsive differential equations (4.95), (4.96), (4.97), if the following inequalities

$$\begin{aligned}
&\alpha_0'(t) \leq f(t,\alpha_0(t)) + g(t,\beta_0(t)),\\
&\beta_0'(t) \geq f(t,\beta_0(t)) + g(t,\alpha_0(t)) \text{ for } t \in [0,T],\ t \neq t_k,\\
&\alpha_0(t_k+0) \leq I_k(\alpha_0(t_k)) + G_k(\beta_0(t_k)),\\
&\beta_0(t_k+0) \geq I_k(\beta_0(t_k)) + G_k(\alpha_0(t_k)),\ \ k = 1,2,\ldots,p,\\
&\alpha_0(0) \leq \alpha_0(T),\ \ \beta_0(0) \geq \beta_0(T)
\end{aligned}$$

hold.

Definition 31. The pair of functions $\alpha_0(t),\beta_0(t) \in PC^1([0,T],\mathbf{R}^N)$ is called a *second type of mixed pair* of lower and upper solution of the periodic boundary value problem for the nonlinear system of impulsive differential equations (4.95), (4.96), (4.97), if the following inequalities

$$\begin{aligned}
&\alpha_0'(t) \leq f(t,\beta_0(t)) + g(t,\alpha_0(t)),\\
&\beta_0'(t) \geq f(t,\alpha_0(t)) + g(t,\beta_0(t)) \text{ for } t \in [0,T],\ t \neq t_k,\\
&\alpha_0(t_k+0) \leq I_k(\beta_0(t_k)) + G_k(\alpha_0(t_k)),\\
&\beta_0(t_k+0) \geq I_k(\alpha_0(t_k)) + G_k(\beta_0(t_k)),\ \ k = 1,2,\ldots,p,\\
&\alpha_0(0) \leq \alpha_0(T),\ \ \beta_0(0) \geq \beta_0(T)
\end{aligned}$$

hold.

We will prove some preliminary results for linear systems of impulsive differential equations.

Let $A = \{a_{ij}\}_{i,j=1}^N$ be a matrix and N be a natural number. We will say that $A > 0$ if $a_{ij} > 0$ for $i,j = 1,2,\ldots,N$.

Definition 32. We will say that matrix $B = \{b_{ij}\}_{i,j=1}^N$ belongs to class Ψ if

P1. $B \geq 0$ and it is irregular;

P2. For $i : 1 \leq i \leq N :\ \ \sum_{j=1}^N b_{ij} \leq 1$.

Definition 33. We will say that matrix $A(t)=\{a_{ij}(t)\}_{i,j=1}^N$ belongs to class Ξ, if

P1. $a_{ij}(t)\in C([0,T],\mathbf{R})$, $a_{ij}(t)\geq 0$ for $j\neq i, i,j=1,2,\dots,N$, $t\in[0,T]$;

P2. $\sum_{j\neq i}a_{ij}(t)+a_{ii}(t)<0, t\in[0,T],\ i=1,2,\dots,N.$

We will define the following operation between vectors:

Definition 34. Let $x=(x_1,x_2,\dots,x_N),\ \ y=(y_1,y_2,\dots,y_N)$.
Then $x@y=(x_1y_1,x_2y_2,\dots,x_Ny_N)$.

We will use the following notation $e=(1,1,\dots,1)$.

For our further investigations we will need the following result for linear systems of impulsive differential inequalities.

Lemma 4.3.1. *Assume that*

1. *Matrix* $A(t)=\{a_{ij}(t)\}_{i,j=1}^N$ *belongs to class* Ξ.
2. *Matrices* $B_k=\{b_{ij}^{(k)}\}_{i,j=1}^N$, $k=1,,\dots p$ *belong to class* Ψ.
3. *Function* $m\in PC^1([0,T],\mathbf{R}^N)$ *satisfies the following inequalities*

$$m'(t)\leq A(t)m(t)\quad t\in[0,T],\ t\neq t_k, \tag{4.98}$$

$$m(t_k+0)\leq B_k(m(t_k)),\ \ k=1,2,\dots,p, \tag{4.99}$$

$$m(0)\leq m(T). \tag{4.100}$$

Then $m(t)\leq 0$ *for* $t\in[0,T]$.

Proof. Consider numbers

$$\varepsilon_{k+1}=\max_{1\leq i\leq N}\max_{t\in(t_k+0,t_{k+1}]}m_i(t)>0,\ \ k=0,1,2,\dots,p.$$

Case 1. Let $\varepsilon_k>0$ for $k=1,2,\dots,p+1$.

Case 1.1. There exists an integer $k:0\leq k\leq p$ such that $m_{j_k}(\xi_k)=\varepsilon_{k+1}$ for a natural number $j_k:\ 1\leq j_k\leq N$ and a point $\xi_k\in(t_k+0,t_{k+1}]$. Then the inequality

$$m'_{j_k}(\xi_k)=\lim_{h\to 0+}\frac{m_{j_k}(\xi_k-h)-m_{j_k}(\xi_k)}{-h}\geq 0$$

holds.

From inequality (4.98) we obtain

$$0\leq m'_{j_k}(\xi_k)\leq\sum_{l\neq j_k}a_{j_kl}(\xi_k)m_l(\xi_k)+a_{j_kj_k}(\xi_k)m_{j_k}(\xi_k)$$

$$\leq\Big(\sum_{l\neq j_k}a_{j_kl}(\xi_k)+a_{j_kj_k}(\xi_k)\Big)\varepsilon_{k+1}<0. \tag{4.101}$$

The obtained contradiction proves the impossibility of this case.

Case 1.2. For all integers $k:0\leq k\leq p$ there exists a natural number $j_k:\ 1\leq j_k\leq N$ such that

$$\lim_{t\to t_k+0}m_{j_k}(t)=\varepsilon_{k+1}$$

and $m_i(t) < \varepsilon_{k+1}$ for $t \in (t_k, t_{k+1}], i = 1, 2, \dots, N$. Then from the jump condition (4.99) we obtain

$$\varepsilon_{k+1} = m_{j_k}(t_k + 0) \leq \sum_{i=1}^{N} b_{j_k i}^{(k)} m_i(t_k) < (\sum_{i=1}^{N} b_{j_k i}^{(k)})\varepsilon_k \leq \varepsilon_k.$$

Using mathematical induction we prove that

$$m_{j_0}(T) < \varepsilon_{p+1} \leq \varepsilon_p \leq \cdots \leq \varepsilon_1 = m_{j_0}(0).$$

The last inequality contradicts inequality (4.100).

Therefore, case 1 is impossible.

Case 2. Let there exists a natural number $l: \quad 1 \leq l \leq p+1$ such that $\varepsilon_l \leq 0$. Let $k = max\{l : \varepsilon_l \leq 0\}$.

If $k = p+1$ then $m(T) \leq 0$.

If $k < p+1$ then $\varepsilon_{k+1} > 0$. According to jump condition (4.99) $m(t_k+0) \leq B_k m(t_k) \leq 0$. Therefore, there exists a natural number $j_k: \ 1 \leq j_k \leq N$ and a point $\xi_k \in (t_k, t_{k+1}]$ such that $m_{j_k}(\xi_k) = \varepsilon_{k+1}$ and $m'_{j_k}(\xi_k) \geq 0$. From inequality (4.98) follows inequality (4.101). The obtained contradiction proves that $k = p+1$. Therefore, $m(T) \leq 0$ and from periodic condition (4.100) we obtain $m(0) \leq 0$. As in the proof above, we obtain that $\varepsilon_l \leq 0$ for $l = 1, 2, \dots, p$. Therefore, $m(t) \leq 0$ on $[0, T]$. □

As a corollary of Lemma 4.3.1 we obtain the following result, that is necessary for the proof of uniqueness and existence of the solution of the periodic boundary value problem for the homogeneous linear impulsive system.

Corollary 4.3.8. *Let conditions* 1 *and* 2 *of Lemma* 4.3.1 *be satisfied.*

Then the periodic boundary value problem for the homogeneous linear system of impulsive differential equations

$$x'(t) = A(t)x(t), \quad t \in [0, T],\ t \neq t_k, \tag{4.102}$$

$$x(t_k + 0) = B_k x(t_k), \tag{4.103}$$

$$x(0) = x(T) \tag{4.104}$$

has only the trivial solution.

We will need some known results for systems of impulsive differential equations (for more details see [15], [89]). Let $U_k(t, s)$ be the fundamental matrix of the linear system of ordinary differential equations

$$x' = A(t)x(t) \quad \text{for} \quad t \in (t_k, t_{k+1}].$$

Then the solution of the linear system of impulsive equations (4.102), (4.103) with the initial condition $x(0) = x_0$ is given by the equality $x(t) = W(t, 0)x_0$, where

$$W(t,s) = \begin{cases} U_k(t,s) & \text{for } \ t,s \in (t_{k-1}, t_k] \\ U_{k+1}(t,t_k)B_k U_k(t_k,s) & \text{for } \ t_{k-1} < s \leq t_k < t \leq t_{k+1} \\ U_{k+1}(t,t_k)(\prod_{j=i+1}^{k} B_j)U_j(t_j,t_{j-1})B_i U_i(t_i,s) & \\ & \text{for } \ t_{i-1} < s \leq t_i < t_k < t \leq t_{k+1}. \end{cases} \tag{4.105}$$

Lemma 4.3.2. *Let* $A(t)=\{a_{ij}(t)\}_{i,j=1}^{N}\in\Xi$ *and* $B_k=\{b_{ij}^{(k)}\}_{i,j=1}^{N}\in\Psi$, $k=1,,\dots p$.
Then $det(E-W(T,0))\neq 0$, *where* E *is the identity* $N\times N$ *matrix.*

Proof. Let $m(t)$ be a solution of the linear system of impulsive equations (4.102), (4.103) with the initial condition $x(0)=x_0$. If $x_0=m(0)=m(T)$ then $x_0=W(T,0)x_0$ and $(E-W(T,0))x_0=0$. According to Corollary 8 the periodic boundary value problem for the linear system of impulsive equations (4.102), (4.103), (4.104) has only the trivial solution, i.e. $x_0=0$ and therefore $det(E-W(T,0))\neq 0$. □

Consider the periodic boundary value problem for the non-homogeneous linear system of impulsive equations

$$x'(t)=A(t)x(t)+h(t)\quad t\in[0,T],\ t\neq t_k, \tag{4.106}$$

$$x(t_k+0)=B_kx(t_k)+\sigma_k, \tag{4.107}$$

$$x(0)=x(T). \tag{4.108}$$

Lemma 4.3.3 (Theorem 2.5.1 [89]). *Let matrix* $(E-W(T,0))$ *be irregular and function* $h\in PC^1([0,T],\mathbf{R}^N)$.
Then the periodic boundary value problem for the linear system of impulsive equations (4.106), (4.107), (4.108) *has a unique solution* $m(t)$, *defined by*

$$m(t)=W(t,0)m_0+\int_0^t W(t,s)h(s)ds+\sum_{0<t_k<t}W(t,t_k+0)\sigma_k, \tag{4.109}$$

where

$$m_0=(E-W(T,0))^{-1}\Big(\int_0^T W(T,s)h(s)ds+\sum_{k=1}^{p}W(T,t_k+0)\sigma_k\Big), \tag{4.110}$$

and $W(t,s)$ *is defind by equality (4.105).*

We will need the following comparison result.

Lemma 4.3.4. *Assume that*
1. *Matrix* $A(t)=\{a_{ij}(t)\}_{i,j=1}^{N}$ *belongs to class* Ξ.
2. *Matrices* B_k, $k=1,,\dots p$ *belong to class* Ψ *and* σ_k *are constants.*
3. *Function* $h\in PC([0,T],\mathbf{R^N})$.
4. *Functions* $v(t)$ *and* $w(t)$ *are lower and upper solutions of the periodic boundary value problem for the linear non-homogeneous system* (4.106), (4.107), (4.108).

Then for $t\in[0,T]$ *the inequality*

$$v(t)\leq W(t,0)m_0+\int_0^t W(t,s)h(s)ds+\sum_{0<t_k<t}W(t,t_k+)\sigma_k\leq w(t), \tag{4.111}$$

holds, where m_0 *is defined by* (4.110), *and* $W(t,s)$ *is defined by* (4.105).

Proof. Consider functions $p(t) = v(t) - x(t)$ and $q(t) = x(t) - w(t)$, where $x(t)$ is the solution of (4.106), (4.107), (4.108). Both functions satisfy the linear systems of inequalities (4.98), (4.99), (4.100) and according to Lemma 4.3.1 the functions are nonpositive on $[0, T]$ which proves Lemma 4.3.4. □

We will prove the method of quasilinearization for approximate obtaining of a solution of the periodic boundary value problem for the systems of nonlinear impulsive differential equations (4.95), (4.96), (4.97). We will prove that the convergence of the successive approximations is quadratic.

Theorem 4.3.1. *Let the following conditions hold:*

1. *The pair of functions* $\alpha_0(t), \beta_0(t) \in PC^1([0,T], \mathbf{R}^N)$, $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$ *is a first type of mixed pair of lower and upper solution of the periodic boundary value for the nonlinear system of impulsive differential equations* (4.95), (4.96), (4.97).

2. *Derivatives* f_x, g_x *exist and they are continuous on* $\Omega(\alpha_0, \beta_0)$, *function* $f_x(t,x)$ *is nondecreasing in x, function* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$ *and* $g_x(t, \alpha_0(t)) \leq 0$, *and for* $x \geq y$

$$f_x(t,x) - f_x(t,y) \leq S_1||x-y||, \;\; g_x(t,y) - g_x(t,x) \leq S_2||x-y||,$$

where $S_1 = \{S_{ij}^{(1)}\}_{i,j=1}^N > 0, S_2 = \{S_{ij}^{(2)}\}_{i,j=1}^N > 0$ *are constant matrices, and* $||.||$ *is a norm in* $\mathbf{R}^N$.

3. *Functions* $I_k, G_k \in C^1(\Gamma_k(\alpha_0, \beta_0), \mathbf{R}^N)$, $I_k'(x)$ *are nondecreasing, functions* $G_k'(x)$ *are nonincreasing,* $k = 1, 2, \ldots, p$ *and the derivatives satisfy the inequalities* $I_k'(\alpha_0(t_k)) \geq 0$, $G_k'(\alpha_0(t_k)) \leq 0$ *and for* $x \geq y$ *the functions satisfy the inequalities*

$$I_k'(x) - I_k'(y) \leq L_k||x-y||, \qquad G_k'(y) - G_k'(x) \leq M_k||x-y||,$$

where $L_k > 0, M_k > 0,\ k = 1, 2, \ldots, p$ *are constant matrices.*

4. *Function* $f_x(t, \alpha_0(t))x$ *is quasimonotone nondecreasing in x, and function* $(f_x(t, \beta_0) - g_x(t, \beta_0))e@x$ *is decreasing in x on the interval* $[0,T]$.

5. *Inequalities* $\Big(I_k'(\beta_0(t_k)) - G_k'(\beta_0(t_k))\Big)e \leq e$ *hold.*

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a/ The sequences are increasing and decreasing, correspondingly;

b/ Both sequences converge uniformly to the unique solution of the periodic boundary value problem for the nonlinear system of impulsive differential equations (4.95), (4.96), (4.97) *in* $S(\alpha_0, \beta_0)$ *for* $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p$, $t_0 = 0$, $t_{p+1} = T$;

c/ The convergence of both sequences is quadratic, i.e. there exists a number $\lambda > 0$ *such that*

$$|||r_{n+1}(t)||| \leq \lambda |||r_n(t)|||^2,$$

where

$$|||r(t)||| = \sup_{t\in[0,T]} ||r(t)||,$$

$$r_{n+1}(t) = \begin{pmatrix} u(t) - \alpha_{n+1}(t) \\ \beta_{n+1}(t) - u(t) \end{pmatrix}.$$

Proof. Consider the periodic boundary value problem for the following system of linear impulsive differential equations

$$\begin{gathered}x'(t) = f(t,\alpha_0(t)) + g(t,\beta_0(t)) + f_x(t,\alpha_0)(x-\alpha_0) + g_x(t,\alpha_0)(y-\beta_0)\\ y'(t) = f(t,\beta_0(t)) + g(t,\alpha_0(t)) + f_x(t,\alpha_0)(y-\beta_0) + g_x(t,\alpha_0)(x-\alpha_0)\\ \text{for } t\in[0,T], t\neq t_k,\end{gathered}$$

$$\begin{aligned}x(t_k+0) &= I_k(\alpha_0(t_k)) + G_k(\beta_0(t_k)) + I_k'(\alpha_0(t_k))[x(t_k)-\alpha_0(t_k)]\\ &\quad + G_k'(\alpha_0(t_k))[y(t_k)-\beta_0(t_k)],\\ y(t_k+0) &= I_k(\beta_0(t_k)) + G_k(\alpha_0(t_k)) + I_k'(\alpha_0(t_k))[y(t_k)-\beta_0(t_k)]\\ &\quad + G_k'(\alpha_0(t_k))[x(t_k)-\alpha_0(t_k)],\ k=1,2,\dots,p,\end{aligned}$$

$$x(0) = x(T), \quad y(0) = y(T). \tag{4.112}$$

The periodic boundary value problem for linear system (4.112) could be written in the vector form

$$p' = A^{(0)}(t)p(t) + h^{(0)}(t) \text{ for } t\in[0,T], t\neq t_k, \tag{4.113}$$

$$p(t_k+0) = B_k^{(0)}p(t_k) + \sigma_k^{(0)},\ k=1,2,\dots,p, \tag{4.114}$$

$$p(0) = p(T), \tag{4.115}$$

where

$$p = \begin{pmatrix} x \\ y \end{pmatrix},$$

$$A^{(0)}(t) = \begin{pmatrix} f_x(t,\alpha_0) & g_x(t,\alpha_0) \\ g_x(t,\alpha_0) & f_x(t,\alpha_0) \end{pmatrix}, \tag{4.116}$$

$$B_k^{(0)} = \begin{pmatrix} I_k'(\alpha_0(t_k)) & G_k'(\alpha_0(t_k)) \\ G_k'(\alpha_0(t_k)) & I_k'(\alpha_0(t_k)) \end{pmatrix}, \tag{4.117}$$

$$h^{(0)}(t) = \begin{pmatrix} f(t,\alpha_0(t)) + g(t,\beta_0(t)) - f_x(t,\alpha_0)\alpha_0 - g_x(t,\alpha_0)\beta_0 \\ f(t,\beta_0(t)) + g(t,\alpha_0(t)) - f_x(t,\alpha_0)\beta_0 - g_x(t,\alpha_0)\alpha_0 \end{pmatrix}, \tag{4.118}$$

$$\sigma_k^{(0)} = \begin{pmatrix} I_k(\alpha_0(t_k)) + G_k(\beta_0(t_k)) - I_k'(\alpha_0(t_k))\alpha_0(t_k) - G_k'(\alpha_0(t_k))\beta_0(t_k) \\ I_k(\beta_0(t_k)) + G_k(\alpha_0(t_k)) - I_k'(\alpha_0(t_k))\beta_0(t_k) - G_k'(\alpha_0(t_k))\alpha_0(t_k) \end{pmatrix}. \tag{4.119}$$

Consider the matrices

$$C^0(t) = \begin{pmatrix} f_x(t,\alpha_0) & -g_x(t,\alpha_0) \\ -g_x(t,\alpha_0) & f_x(t,\alpha_0) \end{pmatrix}, \tag{4.120}$$

$$D_k^0 = \begin{pmatrix} I_k'(\alpha_0(\tau_k)) & -G_k'(\alpha_0(\tau_k)) \\ -G_k'(\alpha_0(\tau_k)) & I_k'(\alpha_0(\tau_k)) \end{pmatrix}. \tag{4.121}$$

From conditions 2, 3, 4 and 5 of Theorem 4.3.1 follows that $C^0(t) \in \Xi$ and $D_k^0 \in \Psi$. According to Lemma 4.3.2 theh inequality $\det(E - C^0(t)) \neq 0$ holds. Therefore, $\det(E - A^0(t)) = \det(E - C^0(t)) \neq 0$. According to Lemma 4.3.3 the periodic boundary value problem for the system of impulsive differential equations (4.113), (4.114), (4.115) has a unique solution given by the equalities (4.109), (4.110). Denote the solution of the periodic boundary value problem (4.113), (4.114), (4.115) by $\alpha_1(t),\ \beta_1(t)$.

We will prove that $\alpha_0(t) \leq \alpha_1(t)$ and $\beta_0(t) \geq \beta_1(t)$ on $[0,T]$. Set $p(t) = \alpha_0(t) - \alpha_1(t),\ q(t) = \beta_1(t) - \beta_0(t)$. From the choice of functions $\alpha_0(t)$, $\beta_0(t)$, $\alpha_1(t)$ and $\beta_1(t)$ we obtain inequalities

$$\begin{aligned} &p' \leq f_x(t,\alpha_0)p - g_x(t,\alpha_0)q,\\ &q' \leq -g_x(t,\alpha_0)p + f_x(t,\alpha_0)q \quad \text{for } t \in [0,T], t \neq t_k, \end{aligned}$$

$$\begin{aligned} &p(t_k+0) \leq I_k'(\alpha_0(t_k))p(t_k) - G_k'(\alpha_0(t_k))q(t_k),\\ &q(t_k+0) \leq I_k'(\alpha_0(t_k))q(t_k) - G_k'(\alpha_0(t_k))p(t_k),\ k = 1,2,\ldots,p, \end{aligned}$$

$$p(0) \leq p(T), \quad q(0) \leq q(T). \tag{4.122}$$

The impulsive inequalities (4.122) can be written in vector form

$$\begin{aligned} &m'(t) \leq C^0(t)m(t) \text{ for } t \in [0,T], t \neq t_k,\\ &m(t_k+0) \leq D_k^0 m(t_k),\ k = 1,2,\ldots,p, \qquad (4.123)\\ &m(0) \leq m(T), \end{aligned}$$

where $m = (p,q)^T$ and matrices $C^0(t)$ and D_k^0 are defined by the equalities (4.123), (4.124).

From conditions 2, 3, 4 and 5 of Theorem 4.3.1 follows that the conditions of Lemma 4.3.1 are satisfied for $2N$ and therefore $m(t) \leq 0$ on $[0,T]$, i.e. $\alpha_0(t) \leq \alpha_1(t)$ and $\beta_1(t) \leq \beta_0(t)$ on $[0,T]$.

We will prove that $\alpha_1(t) \leq \beta_1(t)$. Set $p(t) = \alpha_1(t) - \beta_1(t)$. Then from the choice of the functions $\alpha_1(t)$ and $\beta_1(t)$, equalities (4.113), (4.114), (4.115), and conditions 2, 3 of Lemma 4.3.5 we obtain

$$\begin{aligned} &p' \leq [f_x(t,\alpha_0) - g_x(t,\alpha_0)]p \text{ for } t \in [0,T], t \neq t_k,\\ &p(t_k+0) \leq [I_k'(\alpha_0(t_k) - G_k'(\alpha_0(t_k))]p(\tau_k), \qquad (4.124)\\ &p(0) \leq p(T). \end{aligned}$$

From Lemma 4.3.1 for $A(t) = f_x(t,\alpha_0) - g_x(t,\alpha_0)$, $B_k = I_k'(\alpha_0(t_k) - G_k'(\alpha_0(t_k))$ follows the validity of inequality $p(t) \leq 0$ on $[0,T]$.

Assume that for some natural number m functions $\alpha_m(t)$ and $\beta_m(t)$ are constructed such that $\alpha_{m-1}(t) \leq \alpha_m(t) \leq \beta_m(t) \leq \beta_{m-1}(t)$.

Consider the periodic boundary value problem for the system of linear impulsive differential equations

$$\begin{aligned} x'(t) &= f(t,\alpha_m(t)) + g(t,\beta_m(t)) + f_x(t,\alpha_m)(x-\alpha_m) + g_x(t,\alpha_m)(y-\beta_m)\\ y'(t) &= f(t,\beta_m(t)) + g(t,\alpha_m(t)) + f_x(t,\alpha_m)(y-\beta_m) + g_x(t,\alpha_m)(x-\alpha_m)\\ &\text{for } t \in [0,T], t \neq t_k, \end{aligned}$$

$$x(t_k+0) = I_k(\alpha_m(t_k)) + G_k(\beta_m(t_k)) + I'_k(\alpha_m(t_k))[x(t_k) - \alpha_m(t_k)]$$
$$+G'_k(\alpha_m(t_k))[y(t_k) - \beta_m(t_k)],$$
$$y(t_k+0) = I_k(\beta_m(t_k)) + G_k(\alpha_m(t_k)) + I'_k(\alpha_m(t_k))[y(t_k) - \beta_m(t_k)]$$
$$+G'_k(\alpha_m(t_k))[x(t_k) - \alpha_m(t_k)],$$

$$x(0) = x(T), \quad y(0) = y(T). \tag{4.125}$$

The above periodic boundary value problem for the system of linear impulsive differential equations can be written in the vector form

$$p'_{m+1} = A^{(m)}(t)p_{m+1}(t) + h^{(m)}(t) \text{ for } t \in [0,T], t \neq t_k, \tag{4.126}$$

$$p_{m+1}(t_k+0) = B_k^{(m)} p_{m+1}(t_k) + \sigma_k^{(m)}, \tag{4.127}$$

$$p_{m+1}(0) = p_{m+1}(T), \tag{4.128}$$

where

$$p_{m+1} = \begin{pmatrix} \alpha_{m+1} \\ \beta_{m+1} \end{pmatrix},$$

and the matrices $A^{(m)}(t)$, $B_k^{(m)}$, $h^{(m)}(t)$, $\sigma_k^{(m)}$ are defined by the equalities (4.119) - (4.122), where functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by functions $\alpha_m(t)$ and $\beta_m(t)$ correspondingly.

Consider matrices $C^{(m)}(t)$ and $D_k^{(m)}$, defined by equalities (4.123) and (4.124), where functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by functions $\alpha_m(t)$ and $\beta_m(t)$, respectively.

From inequalities $\alpha_{m-1}(t) \leq \alpha_m(t)$, the monotonicity of derivatives f_x, g_x, and conditions 4, 5 of Theorem 4.3.1 follows that $C^{(m)}(t) \in \Xi$ and $D_k^{(m)} \in \Psi, k = 1, 2, \ldots, p$. Therefore, according to Lemma 4.3.2 the inequality $\det(E - C^{(m)}(t)) \neq 0$ holds and $\det(E - A^{(m)}(t)) = \det(E - C^{(m)}(t)) \neq 0$. According to Lemma 4.3.3 the periodic boundary value problem for system of impulsive differential equations (4.126), (4.127), (4.128) has a unique solution $\alpha_{m+1}(t), \beta_{m+1}(t)$.

As in the proof above for functions $\alpha_1(t)$ and $\beta_1(t)$, we can prove that inequalities $\alpha_m(t) \leq \alpha_{m+1}(t) \leq \beta_{m+1}(t) \leq \beta_m(t)$ hold.

We will prove the convergence of sequences $\{\alpha_m(t)\}_0^\infty$ and $\{\beta_m(t)\}_0^\infty$.

Consider interval $[0,t_1]$. Functions α_m, β_m satisfy equalities (4.126), (4.127), (4.128) and therefore the integral equality

$$p_{m+1}(t) = p_{m+1}(0) + \int_0^t \Big(A^{(m)}(s)p_{m+1}(s) + h^{(m)}(s)\Big)ds \tag{4.129}$$

is satisfied.

Sequences $\{\alpha_m(t)\}$ and $\{\beta_m(t)\}$ are uniformly bounded and equi-continuous on $[0,t_1]$ and therefore the sequences are uniformly convergent on this interval. Set

$$lim_{m\to\infty}\alpha_m(t) = u^{(1)}(t), \quad lim_{m\to\infty}\beta_m(t) = v^{(1)}(t), \ \ t \in [0,t_1]. \tag{4.130}$$

From the definition of functions $\alpha_m(t)$ $\beta_m(t)$ and the uniform convergence follows that $\alpha_0(t) \leq u^{(1)}(t) \leq v^{(1)}(t) \leq \beta_0(t), \ t \in [0,t_1]$.

Taking the limit of equations (4.129) we obtain that the following equations are satisfied for $t \in [0, t_1]$

$$w^{(1)}(t) = w^{(1)}(0) + \int_0^t \Big(A_1(s)w^{(1)}(s) + h_1(s)\Big)ds, \tag{4.131}$$

where

$$w^{(1)} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix},$$

and the matrices $A_1(t) = lim_{m\to\infty}A^{(m)}(t)$ and $h_1(t) = lim_{m\to\infty}h^{(m)}(t)$ are defined by equalities (4.119) and (4.121), where the functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by functions $u^{(1)}(t)$ and $v^{(1)}(t)$ respectively.

From the definition of matrix $A_1(t)$ and function $h_1(t)$ we obtain

$$A_1(t)w^{(1)}(t) + h_1(t) = \begin{pmatrix} f(t, u^{(1)}(t)) + g(t, v^{(1)}(t)) \\ f(t, v^{(1)}(t)) + g(t, u^{(1)}(t)) \end{pmatrix}.$$

From the above equality follows the validity of equality $(w^{(1)})' = A_1(t)w^{(1)} + h_1(t)$ for $t \in [0, t_1]$.

Consider interval $[t_1 + 0, t_2]$. From equalities (4.126) and (4.127) follows that the functions α_m, β_m satisfies on the interval $[t_1 + 0, t_2]$ the integral equation

$$p_{m+1}(t) = p_{m+1}(t_1 + 0) + \int_{t_1}^t \Big(A^{(m)}(s)p_{m+1}(s) + h^{(m)}(s)\Big)ds. \tag{4.132}$$

On this interval functions $\{\alpha_m(t)\}_0^\infty$ and $\{\beta_m(t)\}_0^\infty$ are uniformly bounded and equi- continuous and therefore the sequences are uniformly convergent. Set

$$\lim_{m\to\infty} \alpha_m(t) = u^{(2)}(t), \quad \lim_{m\to\infty} \beta_m(t) = v^{(2)}(t), \; t \in [t_1 + 0, t_2]. \tag{4.133}$$

From the uniform convergent and the definition of functions $\alpha_m(t)$ and $\beta_m(t)$ follows the validity of the inequalities $\alpha_0(t) \le u^{(2)}(t) \le v^{(2)}(t) \le \beta_0(t)$ on $[t_1 + 0,\ t_2]$.

Taking a limit of integral equations (4.127) and (4.132) we obtain the equality $w^{(2)}(t_1) = B_1 w^{(1)}(t_1)$ and

$$\begin{aligned} w^{(2)}(t) &= w^{(2)}(t_1 + 0) + \int_{t_1}^t \Big(A_2(s)w^{(2)}(s) + h_2(s)\Big)ds \\ &= B_1 w^{(1)}(t_1) + \sigma_1 + \int_{t_1}^t \Big(A_2(s)w^{(2)}(s) + h_2(s)\Big)ds, \end{aligned} \tag{4.134}$$

where

$$w^{(2)} = \begin{pmatrix} u^{(2)} \\ v^{(2)} \end{pmatrix},$$

the matrices $A_2(t) = lim_{m\to\infty}A^{(m)}(t)$ and $h_2(t) = lim_{m\to\infty}h^{(m)}(t)$ are defined by equalities (4.116) and (4.118), where the functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by the functions $u^{(2)}$ and $v^{(2)}$ respectively, and the matrices $B_1 = lim_{m\to\infty}B_1^{(m)}$ and σ_1 are defined by equalities

(4.117) and (4.119), where functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by functions $u^{(1)}(t)$ and $v^{(1)}(t)$, respectively.

By induction we prove that on each interval $[t_k+0, t_{k+1}]$, $(k = 0,1,2,\dots,p)$ sequences $\{\alpha_m(t)\}_0^\infty$ and $\{\beta_m(t)\}_0^\infty$ are uniformly convergent. We denote their limits by $u^{(k+1)}(t)$ and $v^{(k+1)}(t)$.

Then $u^{(k+1)}, v^{(k+1)} \in S(\alpha_0,\beta_0)$, $u^{(k+1)}(t) \le v^{(k+1)}(t)$ on $[t_k,t_{k+1}]$, $w^{(k+1)}(t_k) = B_k w^{(k)}(t_k)+\sigma_k$ and the limits are solutions of the linear integral equation

$$\begin{aligned} w^{(k+1)}(t) &= w^{(k+1)}(t_k) + \int_{t_k}^t \Big(A_{k+1}(s)w^{(k+1)}(s)+h_{k+1}(s)\Big)ds \\ &= B_k w^{(k)}(t_k)+\sigma_k + \int_{t_k}^t \Big(A_{k+1}(s)w^{(k+1)}(s)+h_{k+1}(s)\Big)ds, \end{aligned} \tag{4.135}$$

where

$$w^{(k+1)} = \begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix},$$

the matrices $A_{k+1}(t) = lim_{m\to\infty}A^{(m)}(t)$ and $h_k(t) = lim_{m\to\infty}h^{(m)}(t)$ are defined by (4.116) and (4.118), where functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by the functions $u^{(k+1)}(t)$ and $v^{(k+1)}(t)$, and the matrices $B_k = lim_{m\to\infty}B_k^{(m)}$ and σ_k are defined by (4.117) and (4.119), where functions $\alpha_0(t)$ and $\beta_0(t)$ are substituted by functions $u^{(k)}(t)$ and $v^{(k)}(t)$, respectively.

From the definition of matrices $A_{k+1}(t)$ and B_k and the equalities (4.135) we obtain

$$A_{k+1}(t)w^{(k+1)}(t)+h_{k+1}(t) = \begin{pmatrix} f(t,u^{(k+1)}(t))+g(t,v^{(k+1)}(t)) \\ f(t,v^{(k+1)}(t))+g(t,u^{(k+1)}(t)) \end{pmatrix},$$

$$B_k w^{(k)}(t_k)+\sigma_k = \begin{pmatrix} I_k(u^{(k)}(t_k))+G_k(v^{(k)}(t_k)) \\ G_k(u^{(k)}(t_k))+I_k(v^{(k)}(t_k)) \end{pmatrix}. \tag{4.136}$$

Define the piecewise continuous functions $u,v \in PC([0,T],R^n)$ by equalities $u(t) = u^{(k+1)}(t)$ and $v(t) = v^{(k+1)}(t)$ for $t \in (t_k,t_{k+1}], k = 0,1,2,\dots,p$. From the conditions of the functions $u^{(k+1)}(t)$ and $v^{(k+1)}(t)$ it follows that $u,v \in S(\alpha_0,\beta_0)$ and $u(t) \le v(t)$ on $[0,T]$. Consider function $w = (u,v)$. Therefore, the equalities

$$w(t_k+0) = lim_{t\downarrow t_k} w^{(k+1)}(t) = w^{(k+1)}(t_k) = B_k w^k(t_k)+\sigma_k = B_k w(t_k)+\sigma_k \tag{4.137}$$

hold.

From integral equations (4.135) follows that the function $w(t)$ satisfies the integral equations

$$w(t) = w(t_k+0) + \int_{t_k}^t \Big(A(s)w(s)+h(s)\Big)ds, \;\; t \in [t_k+0, t_{k+1}]. \tag{4.138}$$

From equality (4.135) we obtain $w^{(1)}(0) = w^{(p+1)}(T)$ or

$$w(0) = w(T). \tag{4.139}$$

From equalities (4.136) we obtain that for $k = 0,1,2,\dots,p$ the following equalities are satisfied

$$A_{k+1}(t)w^{(k+1)}(t)+h_{k+1}(t) = \begin{pmatrix} f(t,u(t))+g(t,v(t)) \\ f(t,v(t))+g(t,u(t)) \end{pmatrix},$$

$$B_k w^{(k)}(t_k) + \sigma_k = \begin{pmatrix} I_k(u(t_k)) + G_k(v(t_k)) \\ G_k(u(t_k)) + I_k(v(t_k)) \end{pmatrix}.$$

The last equalities and equalities (4.137), (4.138), (4.139) prove that functions $u(t)$ and $v(t)$ are solutions of the periodic boundary value problem

$$\begin{aligned} &u' = f(t,u(t)) + g(t,v(t)),\\ &v' = f(t,v(t)) + g(t,u(t)) \quad \text{for} \quad t \in [0,T],\ t \neq t_k,\\ &u(t_k+0) = I_k(u(t_k)) + G_k(v(t_k)),\\ &v(t_k+0) = I_k(v(t_k)) + G_k(u(t_k)),\end{aligned}$$

$$u(0) = u(T), \quad v(0) = v(T). \tag{4.140}$$

Consider function $p(t) = v(t) - u(t) \geq 0$. From periodic boundary value problem (4.140) and the properties of the derivatives of functions f, g we obtain that

$$\begin{aligned} p' &= \Big(\int_0^1 f_x(s, \lambda v + (1-\lambda)u)ds\Big)p - \Big(\int_0^1 g_x(s, \lambda v + (1-\lambda)u)ds\Big)p\\ &\leq \Big(f_x(t,\beta_0) - g_x(t,\beta_0)\Big)p,\\ p(t_k+0) &= \Big(I_k'(\beta_0(t_k)) - G_k'(\beta_0(t_k))\Big)p(t_k),\end{aligned}$$

$$p(0) = p(T). \tag{4.141}$$

According to Lemma 4.3.1 for $A(t) = f_x(t,\beta_0) - g_x(t,\beta_0)$, $B_k = I_k'(\beta_0(t_k) - G_k'(\beta_0(t_k))$ we obtain $p(t) \leq 0$ on $[0,T]$. Therefore, $p(t) = 0$, which proves that $u(t) = v(t)$.

We will prove that the convergence is quadratic.

Define functions $p_{n+1}(t) = u(t) - \alpha_{n+1}(t)$ and $q_{n+1}(t) = \beta_{n+1}(t) - u(t),\ t \in [0,T]$. For $t \in [0,T], t \neq t_k$ the following inequalities are satisfied

$$\begin{aligned} p_{n+1}' \quad &\leq \quad f(t,u) - f(t,\alpha_n) + g(t,u) - g(t,\beta_n)\\ &\quad\quad - f_x(t,\alpha_n)(p_n - p_{n+1}) - g_x(t,\alpha_n)(q_{n+1} - q_n)\\ &= \quad \Big(\int_0^1 f_x(t, \lambda u + (1-\lambda)\alpha_n)d\lambda\Big)p_n\\ &\quad\quad - \Big(\int_0^1 g_x(t, \lambda u + (1-\lambda)\beta_n)d\lambda\Big)q_n\\ &\quad\quad - f_x(t,\alpha_n)(p_n - p_{n+1}) - g_x(t,\alpha_n)(q_{n+1} - q_n)\\ &\leq \quad (f_x(t,u) - f_x(t,\alpha_n))p_n + (g_x(t,\alpha_n) - g_x(t,\beta_n))q_n\\ &\quad\quad + f_x(t,\alpha_n)p_{n+1} - g_x(t,\alpha_n)q_{n+1}\\ &\leq \quad f_x(t,\alpha_n)p_{n+1} - g_x(t,\alpha_n)q_{n+1} + S_1||p_n||p_n + S_2||\alpha_n - \beta_n||q_n\\ &\leq \quad f_x(t,\alpha_n)p_{n+1} - g_x(t,\alpha_n)q_{n+1}\end{aligned}$$

$$+(\xi + \frac{1}{2}\eta)||p_n||^2 + \frac{3}{2}\eta||q_n||^2, \tag{4.142}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n),\ \ \eta = (\eta_1, \eta_2, \ldots, \eta_n)$,

$$\xi_i = \sum_{j=1}^{n} S_{ij}^1,\ \ \eta_i = \sum_{j=1}^{n} S_{ij}^2,\ \ i = 1, 2, \ldots, n.$$

From the definition of functions $\alpha_n(t)$, $\beta_n(t)$ and the fact that $u(t)$ is a solution of the periodic boundary value problem for the system of impulsive equations (4.95), (4.96), (4.97) we obtain

$$\begin{aligned} q'_{n+1} &\leq f(t,\beta_n) - f(t,u) + g(t,\alpha_n) - g(t,u) \\ &\quad - f_x(t,\alpha_n)(q_n - q_{n+1}) - g_x(t,\alpha_n)(p_{n+1} - p_n) \\ &= \Big(\int_0^1 f_x(t,\lambda\beta_n + (1-\lambda)u)d\lambda\Big) q_n \\ &\quad -\Big(\int_0^1 g_x(t,\lambda\alpha_n + (1-\lambda)u)d\lambda\Big) p_n \\ &\quad - f_x(t,\alpha_n)(q_n - q_{n+1}) - g_x(t,\alpha_n)(p_{n+1} - p_n) \\ &\leq (f_x(t,\beta_n) - f_x(t,\alpha_n))q_n + (g_x(t,\alpha_n) - g_x(t,u))p_n \\ &\quad + f_x(t,\alpha_n)q_{n+1} - g_x(t,\alpha_n)p_{n+1} \\ &\leq -g_x(t,\alpha_n)p_{n+1} + f_x(t,\alpha_n)q_{n+1} + S_1||p_n + q_n||q_n + S_2||p_n||p_n \\ &\leq -g_x(t,\alpha_n)p_{n+1} + f_x(t,\alpha_n)q_{n+1} \\ &\qquad + \frac{3}{2}\xi||q_n||^2 + (\eta + \frac{1}{2}\xi)||p_n||^2. \end{aligned} \tag{4.143}$$

For $k = 1, 2, \ldots, p$ from the jump conditions we obtain

$$\begin{aligned} p_{n+1}(t_k+0) &\leq I_k(u(t_k)) - I_k(\alpha_n(t_k)) + G_k(u(t_k)) - G_k(\alpha_k(t_k)) \\ &\quad + I'_k(\alpha_n(t_k))[\alpha_{n+1}(t_k) - \alpha_n(t_k)] \\ &\quad - G_k(\alpha_n(t_k))[\beta_{n+1}(t_k) - \beta_n(t_k)] \\ &\leq I'_k(\alpha_n(t_k))p_{n+1}(t_k) - G_k(\alpha_n(t_k))q_{n+1}(t_k) \\ &\qquad + \mu_1||p_n(\,t_k)||^2 + \mu_2||q_n(t_k)||^2, \end{aligned} \tag{4.144}$$

$$\begin{aligned} q_{n+1}(\,t_k+0) &\leq -G_k(\alpha_n(t_k))p_{n+1}(t_k) + I'_k(\alpha_n(t_k))q_{n+1}(t_k) \\ &\qquad + \rho_1||p_n(t_k)||^2 + \eta_2||q_n(t_k)||^2, \end{aligned} \tag{4.145}$$

where $\mu_i, \rho_i, i = 1, 2$ are constant vectors.

The differential inequalities (4.142), (4.143), (4.144), (4.145) can be written in a vector form as follows

$$\begin{aligned} & r'_{n+1} \leq A(t)r_{n+1} + P||r_n||^2, \quad t \neq t_k, \\ & r_{n+1}(t_k+0) \leq B_k r_{n+1}(t_k) + Q_k||r_n(t_k)||^2, \\ & r_{n+1}(0) = r_{n+1}(T), \end{aligned} \tag{4.146}$$

where

$$r_{n+1} = \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix}, P = \begin{pmatrix} \xi + 2\eta \\ 2\xi + \eta \end{pmatrix}, Q_k = \begin{pmatrix} \mu_1 + \mu_2 \\ \rho_1 + \rho_2 \end{pmatrix},$$

$$A(t) = \begin{pmatrix} f_x(t,\alpha_n) & -g_x(t,\alpha_n) \\ -g_x(t,\alpha_n) & f_x(t,\alpha_n) \end{pmatrix},$$

$$B_k = \begin{pmatrix} I'_k(\alpha_n(t_k)) & -G'_k(\alpha_n(t_k)) \\ -G'_k(\alpha_n(t_k)) & I'_k(\alpha_n(t_k)) \end{pmatrix}.$$

From the monotonicity of functions $f_x(t,x)$ and $g_x(t,x)$, inequalities $\alpha_m(t) \geq \alpha_0(t)$, $\beta_m(t) \leq \beta_0(t)$, and condition 4 of Theorem 4.3.1 follows that condition 1 of Lemma 4.3.2 is satisfied. According to the conditions 3 and 5 of Theorem 4.3.1, matrices B_k are irregular and from set Ψ. According to Lemma 4.3.2 for $t \in [0,T]$ the inequality

$$r_{n+1}(t) \leq W(t,0)x_0 + \int_0^t W(t,s)P||r_n||^2 ds + \sum_{0<t_k<t} W(t,t_k+0)Q_k||r_n(t_k)||^2 \qquad (4.147)$$

holds, where

$$\begin{aligned} x_0 \quad = \quad & (E - W(T,0))^{-1}\Big(\int_0^T W(T,s)P||r_n||^2 ds \\ & + \sum_{k=1}^{p} W(T,t_k+0)Q_k||r_n(t_k)||^2\Big) \end{aligned} \qquad (4.148)$$

and $W(t,s)$ is defined by equality (4.108).

From inequality (4.147) and equality (4.148) follows that there exists a number $\lambda > 0$ such that $|||r_{n+1}(t)||| \leq \lambda |||r_n(t)|||^2$, where $|||r(t)||| = \sup_{t\in[0,T]} ||r(t)||$.

The above inequality proves the quadratic convergence. □

The next theorem deals with the case when the lower and upper solutions are completely opposite to those in Theorem 4.3.1.

Theorem 4.3.2. *Let the following conditions hold:*

1. *The pair of functions* $\alpha_0(t), \beta_0(t) \in PC^1([0,T], \mathbf{R}^N)$, $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$ *is a* second type *mixed pair of lower and upper solution of the periodic boundary value problem for the nonlinear system of impulsive differential equations* (4.95), (4.96), (4.97).

2. *Derivatives* f_x, g_x *exist and they are continuous on set* $\Omega(\alpha_0, \beta_0)$, *function* $f_x(t,x)$ *is nondecreasing in x, function* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$, $f_x(t, \beta_0(t)) \leq 0$ *and for* $x \geq y$ *the inequality*

$$f_x(t,x) - f_x(t,y) \leq S_1||x-y||, \;\; g_x(t,y) - g_x(t,x) \leq S_2||x-y||,$$

holds, where $S_1 = \{S^{(1)}_{ij}\}^N_{i,j=1} > 0, S_2 = \{S^{(2)}_{ij}\}^N_{i,j=1} > 0$ *are constant matrices and* $||.||$ *is a norm in* $\mathbf{R}^N$.

3. *Functions* $I_k, G_k \in C^1(\Gamma_k(\alpha_0, \beta_0), \mathbf{R}^N)$, $I'_k(x)$ *are nondecreasing, functions* $G'_k(x)$ *are nonincreasing,* $k = 1,2,\ldots,p$, $G'_k(\beta_0(t_k)) \geq 0$, $I'_k(\beta_0(t_k)) \leq 0$ *and for* $x \geq y$ *the functions satisfy the inequalities*

$$I'_k(x) - I'_k(y) \leq L_k||x-y||, \qquad G'_k(y) - G'_k(x) \leq M_k||x-y||,$$

where $L_k > 0, M_k > 0, (k = 1,2,\ldots,p)$ *are constant matrices.*

4. *Function* $g_x(t, \beta_0(t))x$ *is quasimonotone nondecreasing function in x, and function* $(g_x(t,\alpha_0) - f_x(t,\alpha_0))e@x$ *is strictly decreasing function in x on* $[0,T]$.

5. *Inequality* $\Big(G_k'(\beta_0(t_k))-I_k'(\beta_0(t_k))\Big)e\le e$ *holds.*

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a/The sequences are increasing and decreasing correspondingly;

b/ Both sequences converge uniformly to the unique solution of the periodic boundary value problem for the system of nonlinear impulsive differential equations (4.95), (4.96), (4.97) *in* $S(\alpha_0,\beta_0)$ *for* $t\in(t_k,t_{k+1}]$, $k=0,1,2,\dots,p$, $t_0=0$, $t_{p+1}=T$*;*

c/ The convergence of both sequences is quadratic, i.e. there exists a number $\lambda>0$ *such that*

$$|||r_{n+1}(t)|||\le\lambda|||r_n(t)|||^2,$$

where

$$|||r(t)|||=\sup_{t\in[0,T]}||r(t)||,$$

$$r_{n+1}(t)=\begin{pmatrix} u(t)-\alpha_{n+1}(t)\\ \beta_{n+1}(t)-u(t)\end{pmatrix}.$$

Proof. The proof is similar to the proof of Theorem 4.3.1. The successive approximations $\alpha_{m+1}(t)$ and $\beta_{m+1}(t)$ in this case are the unique solution of the periodic boundary value problem for the system of linear impulsive differential equations

$$\begin{aligned} x'(t)&=f(t,\beta_m(t))+g(t,\alpha_m(t))+g_x(t,\beta_m)(x-\alpha_m)\\ &\quad+f_x(t,\beta_m)(y-\beta_m),\\ y'(t)&=f(t,\alpha_m(t))+g(t,\beta_m(t))+g_x(t,\beta_m)(y-\beta_m)\\ &\quad+f_x(t,\beta_m)(x-\alpha_m)\qquad \text{for } t\in[0,T], t\ne t_k,\end{aligned}$$

$$\begin{aligned} x(t_k+0)&=I_k(\beta_m(t_k))+G_k(\alpha_m(t_k))+G_k'(\beta_m(t_k))[x(t_k)-\alpha_m(t_k)]\\ &\quad+I_k'(\beta_m(t_k))[y(t_k)-\beta_m(t_k)],\\ y(t_k+0)&=I_k(\alpha_m(t_k))+G_k(\beta_m(t_k))+G_k'(\beta_m(t_k))[y(t_k)-\beta_m(t_k)]\\ &\quad+I_k'(\beta_m(t_k))[x(t_k)-\alpha_m(t_k)],\end{aligned}$$

$$x(0)=x(T),\quad y(0)=y(T). \tag{4.149}$$

The periodic boundary value problem (4.149) can be written in the form (4.126)-(4.128), where

$$p_{m+1}=\begin{pmatrix}\alpha_{m+1}\\ \beta_{m+1}\end{pmatrix},$$

$$A^{(m)}(t)=\begin{pmatrix} g_x(t,\beta_m) & f_x(t,\beta_m)\\ f_x(t,\beta_m) & g_x(t,\beta_m)\end{pmatrix},$$

$$B_k^{(m)}=\begin{pmatrix} G_k'(\beta_m(t_k)) & I_k'(\beta_m(t_k))\\ I_k'(\beta_m(t_k)) & G_k'(\beta_m(t_k))\end{pmatrix},$$

$$h^{(m)}(t) = \begin{pmatrix} f(t,\beta_m(t)) + g(t,\alpha_m(t)) - g_x(t,\beta_m)\alpha_m - f_x(t,\beta_m)\beta_m \\ f(t,\alpha_m(t)) + g(t,\beta_m(t)) - g_x(t,\beta_m)\beta_m - f_x(t,\beta_m)\alpha_m \end{pmatrix},$$

$$\sigma_k^{(m)} = \begin{pmatrix} I_k(\beta_m(t_k)) + G_k(\alpha_m(t_k)) - G_k'(\beta_m(t_k))\alpha_m(t_k) - I_k'(\beta_m(t_k))\beta_m(t_k) \\ G_k(\alpha_m(t_k)) + I_k(\beta_m(t_k)) - G_k'(\beta_m(t_k))\beta_m(t_k) - I_k'(\beta_m(t_k))\alpha_m(t_k) \end{pmatrix},$$

$$C^m(t) = \begin{pmatrix} g_x(t,\beta_m) & -f_x(t,\beta_m) \\ -f_x(t,\beta_m) & g_x(t,\beta_m) \end{pmatrix}, \quad D_k^m = \begin{pmatrix} G_k'(\beta_m(t_k)) & -I_k'(\beta_m(t_k)) \\ -I_k'(\beta_m(t_k)) & G_k'(\beta_m(t_k)) \end{pmatrix}. \qquad \square$$

The following theorem deals with the case when the derivative of the lower solution of the periodic boundary value problem (4.95), (4.96), (4.97) does not depend on the upper solution.

Theorem 4.3.3. *Let the following conditions hold:*

1. *Functions* $\alpha_0(t), \beta_0(t) \in PC^1([0,T], \mathbf{R}^N)$, $\alpha_0(t) \leq \beta_0(t)$ *for* $t \in [0,T]$ *are lower and upper solutions of the periodic boundary value problem for the system of nonlinear impulsive differential equations* (4.95), (4.96), (4.97).

2. *The continuous derivatives* f_x, g_x *exist on set* $\Omega(\alpha_0,\beta_0)$, *such that* $f_x(t,x)$ *is nondecreasing in x and* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$ *and for* $x \geq y$ *the inequalities*

$$f_x(t,x) - f_x(t,y) \leq S_1||x-y||, \;\; g_x(t,y) - g_x(t,x) \leq S_2||x-y||,$$

hold where $S_1 = \{S_{ij}^{(1)}\}_{i,j=1}^N > 0, S_2 = \{S_{ij}^{(2)}\}_{i,j=1}^N > 0$ *are constant matrices and* $||.||$ *is a norm in* $\mathbf{R}^N$.

3. *Functions* $I_k, G_k \in C^1(\Gamma_k(\alpha_0,\beta_0), \mathbf{R}^N)$, $I_k'(x)$ *are nondecreasing, functions* $G_k'(x)$ *are nonincreasing,* $k = 1,2,\ldots,p$, $G_k'(\beta_0(t_k)) + I_k'(\beta_0(t_k) \geq 0$, *and for* $x \geq y$ *the inequalities*

$$I_k'(x) - I_k'(y) \leq L_k||x-y||, \qquad G_k'(y) - G_k'(x) \leq M_k||x-y||,$$

hold, where $L_k > 0, M_k > 0 < \; k = 1,2,\ldots,p$ *are constant matrices.*

4. *Function* $\Big(F_x(t,\alpha_0(t)) + g_x(t,\beta_0(t))\Big)x$ *is quasimonotone nondecreasing in x, and function* $\Big(g_x(t,\alpha_0) - f_x(t,\alpha_0)\Big)e@x$ *is strictly decreasing in x on* $[0,T]$.

5. *Inequality* $\Big(G_k'(\alpha_0(t_k)) - I_k'(\beta_0(t_k))\Big)e \leq e$ *holds.*

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a/ The sequences are increasing and decreasing correspondingly;

b/ Both sequences converge uniformly to the unique solution of the periodic boundary value problem for the system of nonlinear impulsive differential equations (4.95), (4.96), (4.97) *in* $S(\alpha_0,\beta_0)$ *for* $t \in (t_k, t_{k+1}]$, $k = 0,1,2,\ldots,p$, $t_0 = 0$, $t_{p+1} = T$*;*

c/ The convergence of both sequences is quadratic, i.e. there exists a number $\lambda > 0$ *such that*

$$|||r_{n+1}(t)||| \leq \lambda |||r_n(t)|||^2,$$

where

$$|||r(t)||| = \sup_{t\in[0,T]} ||r(t)||,$$

$$r_{n+1}(t) = \begin{pmatrix} u(t) - \alpha_{n+1}(t) \\ \beta_{n+1}(t) - u(t) \end{pmatrix}.$$

Proof. For every whole number m we consider the periodic boundary value problem for the system of impulsive linear differential equations

$$\begin{aligned} x'(t) &= f(t,\alpha_n(t)) + g(t,\alpha_n(t)) + \Big(f_x(t,\alpha_n) + g_x(t,\beta_n)\Big)(x-\alpha_n) \\ &\qquad \text{for } t\in[0,T], t\neq t_k, \\ x(t_k+0) &= I_k(\alpha_n(t_k)) + G_k(\alpha_n(t_k)) \\ &\quad + \Big(I_k'(\alpha_n(t_k)) + G_k'(\beta_n(t_k))\Big)[x(t_k)-\alpha_n(t_k)], \\ x(0) &= x(T), \end{aligned} \tag{4.150}$$

and the periodic boundary value problem for the system of impulsive linear differential equations

$$\begin{aligned} y'(t) &= f(t,\beta_n(t)) + g(t,\beta_n(t)) + \Big(f_x(t,\alpha_n) + g_x(t,\beta_n)\Big)(y-\beta_n) \\ &\qquad \text{for } t\in[0,T], t\neq t_k, \\ y(t_k+0) &= I_k(\beta_n(t_k)) + G_k(\beta_n(t_k)) \\ &\quad + \Big(I_k'(\alpha_n(t_k)) + G_k'(\beta_n(t_k))\Big)[y(t_k)-\beta_n(t_k)], \\ y(0) &= y(T). \end{aligned} \tag{4.151}$$

According to Lemma 4.3.3 the periodic boundary value problems (4.150) and (4.151) have unique solutions $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ for every fixed pair of functions $\alpha_n(t)$ and $\beta_n(t)$.

Let $n=0$. We will prove that $\alpha_0(t)\leq\alpha_1(t)$ and $\beta_0(t)\geq\beta_1(t)$ on the interval $[0,T]$. Set $p(t)=\alpha_0(t)-\alpha_1(t)$, $q(t)=\beta_1(t)-\beta_0(t)$. Therefore,, we have the following two periodic boundary value problems for functions $p(t)$ and $q(t)$:

$$\begin{aligned} p' &\leq \Big(f_x(t,\alpha_0) + g_x(t,\beta_0)\Big)p \quad \text{for } t\in[0,T], t\neq t_k, \\ p(t_k+0) &\leq \Big(I_k'(\alpha_0(t_k)) + G_k'(\beta_0(t_k))\Big)p(t_k), \\ p(0) &\leq p(T), \end{aligned}$$

and

$$\begin{aligned} q' &\leq \Big(g_x(t,\beta_0) + f_x(t,\beta_0)\Big)q \quad \text{for } t\in[0,T], t\neq t_k, \\ q(t_k+0) &\leq \Big(I_k'(\beta_0(t_k)) + G_k'(\beta_0(t_k))\Big)q(t_k), \\ q(0) &\leq q(T). \end{aligned}$$

According to Lemma 4.3.1 inequalities $p(t)\leq 0$ and $q(t)\leq 0$ hold on $[0,T]$, i.e. $\alpha_0(t)\leq\alpha_1(t)$ and $\beta_1(t)\leq\beta_0(t)$ on $[0,T]$.

We will prove that $\alpha_1(t)\leq\beta_1(t)$. Set $p(t)=\alpha_1(t)-\beta_1(t)$. From the periodic boundary value problems (4.150) and (4.151), conditions 2, 3, 4 and 5 of Theorem 4.3.3 and Lemma 4.3.2 we obtain

$$\begin{aligned} &p' \leq [f_x(t,\alpha_0) + g_x(t,\beta_0)]p \text{ for } t\in[0,T], t\neq t_k, \\ &p(t_k+0) \leq [I_k'(\alpha_0(t_k) + G_k'(\beta_0(t_k))]p(t_k),\ k=1,2,\dots,p, \\ &p(0) \leq p(T). \end{aligned}$$

According to Lemma 4.3.1 inequality $p(t) \leq 0$ holds on $[0,T]$, i.e. $\alpha_1(t) \leq \beta_1(t)$.

As in the proof of Theorem 4.3.1, we obtain two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$, $\alpha_n, \beta_n \in S(\alpha_{n-1}, \beta_{n-1})$, $\alpha_n(t) \leq \beta_n(t), t \in [0,T]$, which are uniformly convergent on the intervals $(t_k, t_{k+1}]$, $k = 0, 1, \ldots, p$.

Let their limits are $u(t)$ and $v(t)$ correspondingly.

From the uniform convergence and the definition of the functions $\alpha_n(t)$ and $\beta_n(t)$ follows the validity of inequalities

$$\alpha_0(t) \leq u(t) \leq v(t) \leq \beta_0(t).$$

Since functions $\alpha_n(t)$ and $\beta_n(t)$ are solutions of the periodic boundary value problems (4.150) and (4.151), we obtain that functions $u(t)$ and $v(t)$ are solutions of the periodic boundary value problem (4.95), (4.96), (4.97). From the fact that functions f, g and I_k, G_k are Lipschitz, follows that $u(t) = v(t)$ on $[0,T]$.

We will prove that the convergence is quadratic.

Define the functions $p_{n+1}(t) = u(t) - \alpha_{n+1}(t)$ and $q_{n+1}(t) = \beta_{n+1}(t) - u(t),\ t \in [0,T]$. For $t \in [0,T], t \neq t_k$ we obtain the inequalities

$$\begin{aligned}
p'_{n+1} &\leq f(t,u) - f(t,\alpha_n) + g(t,u) - g(t,\alpha_n) \\
&\quad - f_x(t,\alpha_n)(p_n - p_{n+1}) + g_x(t,\beta_n)(p_{n+1} - p_n) \\
&= \Big(\int_0^1 f_x(t, \lambda u + (1-\lambda)\alpha_n) d\lambda\Big) p_n + \Big(\int_0^1 g_x(t, \lambda u + (1-\lambda)\alpha_n) d\lambda\Big) p_n \\
&\quad - f_x(t,\alpha_n)(p_n - p_{n+1}) + g_x(t,\beta_n)(p_{n+1} - p_n) \\
&\leq (f_x(t,u) - f_x(t,\alpha_n)) p_n + (g_x(t,\alpha_n) - g_x(t,\beta_n)) p_n \\
&\quad + f_x(t,\alpha_n) p_{n+1} + g_x(t,\beta_n) p_{n+1} \\
&\leq \Big(f_x(t,\alpha_n) + g_x(t,\beta_n)\Big) p_{n+1} + S_1 ||p_n|| p_n + S_2 ||\alpha_n - \beta_n|| p_n \\
&\leq \Big(f_x(t,\alpha_n) + g_x(t,\beta_n)\Big) p_{n+1} + (\xi + \frac{1}{2}\eta) ||p_n||^2 + \frac{1}{2}\eta ||q_n||^2,
\end{aligned} \tag{4.152}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n),\ \ \eta = (\eta_1, \eta_2, \ldots, \eta_n)$,

$$\xi_i = \sum_{j=1}^n S_{ij}^1,\ \ \eta_i = \sum_{j=1}^n S_{ij}^2,\ \ i = 1, 2, \ldots, n.$$

For $t \in [0,T], t \neq t_k$ from the definition of the functions $\alpha_n(t)$ and $\beta_n(t)$ and the fact that $u(t)$ is a solution of the periodic boundary value problem for the system of nonlinear impulsive differential equations (4.95), (4.96), (4.97), we obtain

$$\begin{aligned}
q'_{n+1} &\leq f(t,\beta_n) - f(t,u) + g(t,\beta_n) - g(t,u) \\
&\quad - f_x(t,\beta_n)(q_n - q_{n+1}) + g_x(t,\beta_n)(q_{n+1} - q_n) \\
&= \Big(\int_0^1 f_x(t, \lambda\beta_n + (1-\lambda)u) d\lambda\Big) q_n + \Big(\int_0^1 g_x(t, \lambda\beta_n + (1-\lambda)u) d\lambda\Big) q_n \\
&\quad - f_x(t,\beta_n)(q_n - q_{n+1}) + g_x(t,\beta_n)(q_{n+1} - q_n) \\
&\leq (g_x(t,u) - g_x(t,\beta_n)) q_n + (f_x(t,\beta_n) + g_x(t,\beta_n) q_{n+1} \\
&\leq (f_x(t,\beta_n) + g_x(t,\beta_n) q_{n+1} + S_2 ||q_n|| q_n
\end{aligned}$$

$$\leq \Big(f_x(t,\beta_n)+g_x(t,\beta_n)\Big)q_{n+1}+(\xi+\frac{3}{2}\eta)||q_n||^2+\frac{1}{2}\eta. \tag{4.153}$$

For $k=1,2,\dots,p$ from the jump conditions we have

$$\begin{aligned} p_{n+1}(t_k+0) &\leq I_k(u(t_k))-I_k(\alpha_n(t_k))+G_k(u(t_k))-G_k(\alpha_n(t_k)) \\ &\quad +\Big(I_k'(\alpha_n(t_k))-G_k(\beta_n(t_k))\Big)[p_{n+1}(t_k)-p_n(t_k)] \\ &\leq \Big(I_k'(\alpha_n(t_k))+G_k'(\beta_n(t_k))\Big)p_{n+1}(t_k) \\ &\quad +\mu_1||p_n(t_k)||^2+\mu_2||q_n(t_k)||^2, \end{aligned} \tag{4.154}$$

$$\begin{aligned} q_{n+1}(t_k+0) \leq \Big(I_k'(\alpha_n(t_k))+G_k'(\beta_n(t_k))\Big)q_{n+1}(t_k) \\ +\eta_1||p_n(t_k)||^2+\eta_2||q_n(t_k)||^2. \end{aligned} \tag{4.155}$$

where $\mu_i,\eta_i, i=1,2$ are constant vectors.

The differential inequalities (4.152) -(4.155) and the periodic boundary value conditions for functions $p(t)$ and $q(t)$ can be written in the following form

$$r_{n+1}'\leq A(t)r_{n+1}+P||r_n||^2, \quad t\neq t_k, \tag{4.156}$$

$$r_{n+1}(t_k+0)\leq B_k r_{n+1}(t_k)+Q_k||r_n(t_k)||^2, \tag{4.157}$$

$$r_{n+1}(0)=r_{n+1}(T), \tag{4.158}$$

where

$$r_{n+1}=\begin{pmatrix} p_{n+1} \\ q_{n+1}\end{pmatrix},$$

$$A(t)=\begin{pmatrix} f_x(t,\alpha_n)+g_x(t,\beta_n) & 0 \\ 0 & f_x(t,\alpha_n)+g_x(t,\beta_n)\end{pmatrix},$$

$$P=\begin{pmatrix} \xi+\frac{1}{2}\eta & \frac{1}{2}\eta \\ \frac{1}{2}\eta & \xi+\frac{3}{2}\eta\end{pmatrix}, Q_k=\begin{pmatrix} \mu_1 & \mu_2 \\ \eta_1 & \eta_2\end{pmatrix},$$

$$B_k=\begin{pmatrix} I_k'(\alpha_n(t_k))+G_k'(\beta_n(t_k)) & 0 \\ 0 & I_k'(\alpha_n(t_k))+G_k'(\beta_n(t_k))\end{pmatrix}.$$

According to Lemma 4.3.2 and inequalities (4.156) - (4.158) follows that for $t\in[0,T]$ inequality

$$\begin{aligned} r_{n+1}(t) \leq W(t,0)x_0+\int_0^t W(t,s)P||r_n||^2 ds \\ +\sum_{0<t_k<t} W(t,t_k+0)Q_k||r_n(t_k)||^2 \end{aligned} \tag{4.159}$$

holds, where

$$x_0 = (E - W(T,0))^{-1}\Big(\int_0^T W(T,s)P||r_n||^2 ds + \sum_{k=1}^{p} W(T,t_k+0)Q_k||r_n(t_k)||^2\Big), \qquad (4.160)$$

and matrix $W(t,s)$ is defined by equality (4.105).

From relations (4.159) and (4.160) follows the existence of a number $\lambda > 0$ such that

$$|||r_{n+1}(t)||| \le \lambda |||r_n(t)|||^2,$$

where $|||r(t)||| = \sup_{t\in[0,T]} ||r(t)||$.

The inequality proves the quadratic convergence of both sequences of successive approximations. □

As partial cases of the proved theorems we can obtain several interesting results for the periodic boundary value problems for systems of nonlinear ordinary differential equations.

Consider the periodic boundary value problem for the system of ordinary differential equations

$$x' = f(t,x(t)) + g(t,x(t)), \quad t \in [0,T], \qquad (4.161)$$

$$x(0) = x(T), \qquad (4.162)$$

where $x \in \mathbf{R}^n$, $f,g : [0,T]\times\mathbf{R}^n \to \mathbf{R}^n$.

Let $\alpha,\beta \in C([0,T],\mathbf{R}^n)$ be such that $\alpha(t) \le \beta(t)$. Consider set

$$CS(\alpha,\beta) = \{u \in C([0,T],\mathbf{R}^n) : \quad \alpha(t) \le u(t) \le \beta(t) \text{ for } t \in [0,T]\}. \qquad (4.163)$$

From Theorems 4.3.1, 4.3.2, and 4.3.3 we obtain the following results:

Theorem 4.3.4. *Let the following conditions hold:*

1. *Functions* $\alpha_0(t),\beta_0(t) \in C([0,T],\mathbf{R}^n)$, $\alpha_0(t) \le \beta_0(t)$ *for* $t \in [0,T]$ *and*

$$\alpha_0'(t) \le f(t,\beta_0(t)) + g(t,\alpha_0(t)),$$
$$\beta_0'(t) \ge f(t,\alpha_0(t)) + g(t,\beta_0(t)) \text{ for } t \in [0,T],$$
$$\alpha_0(0) \le \alpha_0(T), \;\; \beta_0(0) \ge \beta_0(T).$$

2. *Functions* f_x, g_x *exist and they are continuous on* $\Omega(\alpha_0,\beta_0)$, $f_x(t,x)$ *is nondecreasing in x,* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$, $f_x(t,\beta_0(t)) \le 0$ *and for* $x \ge y$

$$f_x(t,x) - f_x(t,y) \le S_1||x-y||, \;\; g_x(t,y) - g_x(t,x)| \le S_2||x-y||$$

where $S_1 > 0, S_2 > 0$ *are constant matrices,* $||.||$ *is a norm in* $\mathbf{R}^n$.

3. *Function* $g_x(t,\beta_0(t))x$ *is quasimonotone nondecreasing in x and the function* $(-f_x(t,\alpha_0) + g_x(t,\alpha_0))e@x$ *is strictly decreasing in x on* $[0,T]$.

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a. The sequences are increasing and decreasing correspondingly;

b. Both sequences uniformly convergent on $[0,T]$ *to the unique solution of the periodic boundary value problem* (4.161), (4.162) *in* $CS(\alpha_0,\beta_0)$*;*

c. The convergence is quadratic.

Theorem 4.3.5. *Let the following conditions hold:*

1. *Functions* $\alpha_0(t), \beta_0(t) \in C([0,T], \mathbf{R}^n)$*,* $\alpha_0(t) \le \beta_0(t)$ *for* $t \in [0,T]$ *and*

$$\alpha_0'(t) \le f(t, \beta_0(t)) + g(t, \alpha_0(t)),$$
$$\beta_0'(t) \ge f(t, \alpha_0(t)) + g(t, \beta_0(t)) \text{ for } t \in [0,T],$$
$$\alpha_0(0) \le \alpha_0(T), \ \ \beta_0(0) \ge \beta_0(T).$$

2. *Functions* f_x, g_x *exist and they are continuous on* $\Omega(\alpha_0, \beta_0)$*,* $f_x(t,x)$ *is nondecreasing in x,* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$*,* $f_x(t, \beta_0(t)) \le 0$ *and for* $x \ge y$ *the inequalities*

$$f_x(t,x) - f_x(t,y) \le S_1 ||x-y||, \ \ g_x(t,y) - g_x(t,x)| \le S_2 ||x-y||$$

hold, where $S_1 > 0, S_2 > 0$ *are constant matrices,* $||.||$ *is a norm in* $\mathbf{R}^n$*.*

3. *Function* $g_x(t, \beta_0(t))x$ *is quasimonotone nondecreasing in x and the function* $(-f_x(t, \alpha_0) + g_x(t, \alpha_0))e@x$ *is strictly decreasing in x on* $[0,T]$*.*

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a. The sequences are increasing and decreasing correspondingly;

b. Both sequences uniformly convergent on $[0,T]$ *to the unique solution of the periodic boundary value problem* (4.161), (4.162) *in* $CS(\alpha_0, \beta_0)$*;*

c. The convergence is quadratic.

Theorem 4.3.6. *Let the following conditions hold:*

1. *Functions* $\alpha_0(t), \beta_0(t) \in C([0,T], \mathbf{R}^n)$*,* $\alpha_0(t) \le \beta_0(t)$ *for* $t \in [0,T]$ *are such that*

$$\alpha_0'(t) \le f(t, \alpha_0(t)) + g(t, \alpha_0(t)),$$
$$\beta_0'(t) \ge f(t, \beta_0(t)) + g(t, \beta_0(t)) \text{ for } t \in [0,T],$$
$$\alpha_0(0) \le \alpha_0(T), \ \ \beta_0(0) \ge \beta_0(T).$$

2. *Functions* f_x, g_x *exist and they are continuous on* $\Omega(\alpha_0, \beta_0)$*,* $f_x(t,x)$ *is nondecreasing in x,* $g_x(t,x)$ *is nonincreasing in x for* $t \in [0,T]$ *and for* $x \ge y$ *inequalities*

$$f_x(t,x) - f_x(t,y) \le S_1 ||x-y||, \ \ g_x(t,y) - g_x(t,x)| \le S_2 ||x-y||$$

hold, where $S_1 > 0, S_2 > 0$ *are constant matrices,* $||.||$ *is a norm in* $\mathbf{R}^n$*.*

3. *Function* $\Big(f_x(t, \alpha_0(t)) + g_x(t, \alpha_0(t))\Big)x$ *is quasimonotone nondecreasing in x and function* $(f_x(t, \alpha_0) + g_x(t, \beta_0))e@x$ *is strictly decreasing in x on* $[0,T]$*.*

Then there exist two sequences of functions $\{\alpha_m(t)\}_0^\infty$ *and* $\{\beta_m(t)\}_0^\infty$ *such that:*

a. The sequences are increasing and decreasing correspondingly;

b. Both sequences uniformly convergent on $[0,T]$ *to the unique solution of the periodic boundary value problem* (4.161), (4.162) *in* $CS(\alpha_0, \beta_0)$*;*

c. The convergence is quadratic.

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Index

H

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

Y